RESEARCH ARTICLE **ISSN 2738-9898 (Print) ISSN 2738-9901 (Online)**

Hypergeometric Function and its Application on Heat Kernel in a Torus Surface with a Rectangular Lattice

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Received : 16 April, 2024; Received in revised form : 11 May 2024; Accepted : 21 May 2024; Published: 30 July ,2024

Abstract

Hypergeometric functions are transcendental functions that are applicable in various branches of mathematics, physics, and engineering. They are solutions to a class of differential equations called hypergeometric differential equations. In this paper we will be using theta functions, null zeta function and Ramanujan's identities to study the behavior of heat kernel in a torus surface with rectangular Lattice.

Keywords: Guass Hypergeometric Function, Theta function, Heat Kernel, Torus

1. Introduction

1.1 Guass Hypergeometric Function

The generalized hypergeometric function ${}_pF_q$ having p number of numerator and q number of denominator parameters is defined by [1- 4] is given by

$$ {}_pF_q\left[\begin{matrix} h_1,\dots,h_p \\ k_1,\dots,k_q \end{matrix};\quad z\right] = \sum_{n=0}^{\infty} \frac{(h_1)_n \dots (h_p)_n}{(k_1)_n \dots (k_q)_n} \frac{z^n}{n!} \qquad \text{... (1.1)} $$

where $h_1, h_2, \dots h_p, k_1, k_2 \dots k_q \in C$ and $(\lambda)_n$ denotes the Pochhammer symbol with its usual representation in terms of Gamma function defined by

$$ (Z)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad \text{... (1.2)} $$

Where $\Gamma(\lambda)$ is the gamma function defined by

$$ \Gamma(\lambda) = \int_0^{\infty} t^{z-1} e^{-t} dt, \qquad \text{...(1.3)} $$

where $\operatorname{Re}(z) > 0$

The generalized hypergeometric series for $p = 2$ and $q = 1$ is reduces to Guassian hypergeometric function defined for $a + b \le c$ and $z \in (0,1)$

$$\text{F(a, b; c; z)} = {}_2F_1\left[\begin{matrix} a & b; \\ & c; \end{matrix} \quad z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad \ldots(1.4)$$

or equivalently[4],

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \quad ;1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \ldots(1.5)$$

The exponential function in terms of hypergeometric function can be written as

$${}_0F_0\left[\begin{matrix} - \\ - \end{matrix} \quad ; x\right] = e^x \qquad \ldots (1.6)$$

Now the Guass formula for the hypergeometric function is given by ${}_2F_1$ (a, b; c; z) in terms of gamma function is

$${}_2F_1\left[\begin{matrix} x, \quad y \\ \frac{1}{2}(x+y+1) \end{matrix} \quad ; \frac{1}{2}\right] = \frac{\sqrt{\pi}\,\Gamma\left(\frac{x}{2}+\frac{y}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)\Gamma\left(\frac{y}{2}+\frac{1}{2}\right)} \qquad \ldots (1.7)$$

1.2 Fourier Sine and Cosine Series

The Fourier sine series and the cosine series for the function $f(x) = e^x$, $0 < x < l$, is given by [5]

$$e^x = \sum_{n=1}^{\infty} \frac{2\pi}{l^2 + n^2\pi^2}\left[1 - e^l(-1)^n\right]\sin\left(\frac{n\pi x}{l}\right) \qquad \ldots (1.8)$$

$$e^x = \frac{1}{l}(e^l - 1) + \sum_{n=1}^{\infty} \frac{2l}{l^2 + n^2\pi^2}\left[e^l(-1)^n - 1\right]\cos\left(\frac{n\pi x}{l}\right) \qquad \ldots (1.9)$$

respectively.

1.3 Theta Function

The theta function [6] for z is $z \in C$ and $q \in C$ with $|q| < 1$, we define

$$\gamma_1(z,q) = \sum_{k \in z} (-1)^{\left(k-\frac{1}{2}\right)} q^{\left(k+\frac{1}{2}\right)^2} e^{(2k+1)\pi i z} \qquad \ldots (1.10)$$

$$\gamma_2(z,q) = \sum_{k \in z} q^{\left(k+\frac{1}{2}\right)^2} e^{(2k+1)\pi i z} \qquad \ldots (1.11)$$

$$\gamma_3(z,q) = \sum_{k \in z} q^{k^2} e^{2k\pi i z} \qquad \ldots (1.12)$$

$$\gamma_4(z,q) = \sum_{k\in z} (-1)^k q^{k^2} e^{2k\pi iz} \qquad \text{... (1.13)}$$

Corollary 1: The relations (1.10-1.13) are valid for the real valued function $q \in (0,1)$.

The relations (1.9-1.12) coincides with the Fourier series with the real coefficients whose values consists of symmetry in the power of k where values proves the symmetry in power of *k*. Then we can write the Jacobi's theta function as the function of the pair of the variable $(z,\tau) \in CxH$ where $H = \{z \in C; \operatorname{Im}(Z) > 0\}$

1.4 Heat Kernel

Heat kernel is the fundamental solution to the heat equation on a given domain under the defined boundary conditions. It represents the evolution of temperature in a region whose boundary is held fixed at a fixed temperature, such that the initial unit of heat energy kept at an initial temperature [7]. The heat kernel in *n*- dimensional Euclidean space follows the Guassian function. [8]

$$K(t,x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{\|x+y\|^2}{4t}} \qquad \text{...(1.14)}$$

2. Main Results

2.1 Elliptical integral and theta function

2.1.1. Theorem

If the theta function for z is $z \in C$ *and* $q \in C$ *with* $|q| < 1$, *The square of the ratio of the theta nulls gives the elliptical modulus or the eccentricity of the ellipse. i. e. .*

$$\frac{[\theta_2(q)]^2}{[\theta_3(q)]^2} = k$$

where k is the elliptical modulus or the eccentricity of the ellipse.

Proof:

If q is replaced by $e^{iz\tau}, \tau \in H$, in (1.9) then the series converges. Now by Whittaker Watson of Steirs and Shakarelin [9], the product representation for $\gamma_3(z,q)$ is represented by

$$\gamma_3(z,q) = \prod_{k\geq 1} (1-q^{2k})(1+q^{2k-1}e^{2\pi iz})\ (1+q^{2k-1}e^{-2\pi iz})$$

$$= \prod_{k\geq 1} (1-q^{2k})(1+q^{2k-1}\cos(2\pi z) + q^{4k-2} \qquad \text{...(2.1)}$$

Now for $q \in (0,1)$ in (2.1) we have,

$$\gamma_3\left(\frac{1}{2} + l, q)\right) < \gamma_3(z,q); \qquad \text{...(2.2)}$$

for every $l \in Z, z \in R$

Also from (2.1), the theta null functions depends upon q are desired by setting $z \geq 0$. But we have $\gamma_1(0,q) = 0; |q| < 1$. As it is an odd function of z and the theta functions are even with respect with respect to z . By setting $z = 0$, from equations (1.9-1.11), the theta nulls [10] are given by

$$\theta_2(q) = \gamma_2(0,q) = \sum_{k \in z} q^{\left(k+\frac{1}{2}\right)^2} \quad ...(2.3)$$

$$\theta_3(q) = \gamma_2(0,q) = \sum_{k \in z} q^{k^2} \quad ...(2.4)$$

$$\theta_4(q) = \gamma_2(0,q) = \sum_{k \in z} (-1)^k q^{k^2} \quad ...(2.5)$$

The relations (2.3-2.5) can be established by using Poisson Summation formulae by setting $q = e^{\pi i \tau}$ as given below.

$$\theta_3(e^{\pi i \tau}) = \sqrt{\frac{i}{2}} \theta_3\left(e^{-\frac{\pi i}{\tau}}\right) \quad ...(2.6)$$

$$\theta_2(e^{\pi i \tau}) = \sqrt{\frac{i}{2}} \theta_4\left(e^{-\frac{\pi i}{\tau}}\right) \quad ...(2.7)$$

And

$$\theta_4(e^{\pi i \tau}) = \sqrt{\frac{i}{2}} \theta_2\left(e^{-\frac{\pi i}{\tau}}\right) \quad ...(2.8)$$

From the Ramanujan identity [11], the Berndt's relation expresses the connection between theta nulls of Jacobi's theta function and hypergeometric function

i.e,

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ & 1 \end{matrix} ; k^2\right] = \theta_3(q)^2 \quad ...(2.9)$$

where k is the elliptical modulus or often called the eccentricity. The numerical value of k is defined by

$$k = \frac{[\theta_2(q)]^2}{[\theta_3(q)]^2} \quad ...(2.10)$$

and $q = e^{\pi i \tau}$

Thus eccentricity depends upon $q = e^{\pi i \tau}$, $\tau \in |H|$. Further let us define an complementary elliptical modulus k' such that

$$k^2 + k'^2 = 1 \quad ...(2.11)$$

Hence a complete elliptical integral of the first kind is obtained which is given by [12,13]

$$K(k) = \int_0^\infty \frac{d\psi}{1 - k^2 \sin(\psi)^2} = \frac{\pi}{2} {}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ & 1 \end{matrix} ; k^2\right] \quad ...(2.12)$$

Further the relationship between θ_2, θ_3 and θ_4 as defined by Conway et. al [14] is

$$\theta_2(q)^4 + \theta_4(q)^4 = \theta_3(q)^4 \qquad ...(2.13)$$

Now, dividing both sides of (2.13) by $\theta_3(q)^4$, we get

$$\frac{\theta_2(q)^4}{\theta_3(q)^4} + \frac{\theta_4(q)^4}{\theta_3(q)^4} = \frac{\theta_3(q)^4}{\theta_3(q)^4}$$

Or

$$\frac{\theta_2(q)^4}{\theta_3(q)^4} + \frac{\theta_4(q)^4}{\theta_3(q)^4} = 1 \qquad ...(2.14)$$

Now comparing (2.14) with (2.11) we get

$$k' = \frac{[\theta_4(q)]^2}{[\theta_3(q)]^2} \qquad ...(2.15)$$

This proves the theorem.

Corollary 2: Since the elliptical integrals can be represented by another parameter n other than the modulus value of k [15]. Then

$$n = k^2 = \frac{[\theta_2(q)]^4}{[\theta_3(q)]^4} \qquad ...(2.16)$$

Corollary 3: For any *n'*, the conjugate of *n*, the relation $n^2 + n'^2 = 1$ satisfies then, we get

$$n' = k'^2 = \frac{[\theta_4(q)]^4}{[\theta_3(q)]^4} \qquad ...(2.17)$$

Now let us introduce the real valued theta functions defined by

$$\theta_j(t) = \theta_j(e^{-\pi t}) \qquad ...(2.18)$$

for $j = 2,3,$ and 4. This is the theta nulls of Jacobi's theta function restricted to $q = e^{-\pi\tau}, \tau \in R^+$

2.2 Effect of Heat Kernel

2.2.1 Theorem

The hottest and coldest temperature in the rectangular lattice of two extremal temperature conditions in a torus surface with a rectangular lattice is given by the hypergeometric function

$$A(1,t) = k'\ {}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ & 1\end{matrix}\ ; k^2\right]$$

a*nd*

$$B(1,t) = {}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ & 1\end{matrix}\ ; 1 - k'^2\right]$$

Proof:

Let us consider the rectangular lattice and subject to precise coldest and hottest temperature on the rectangular torus within a fixed area as shown in the figure 1 and figure 2.

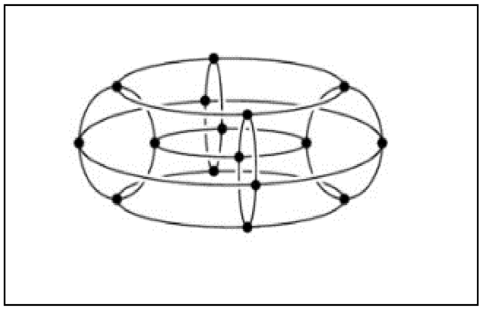

Figure 1: Torus surface

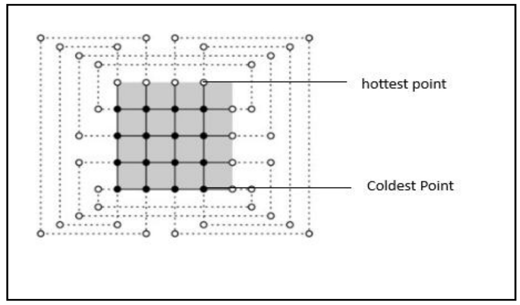

Figure 2. Rectangular lattice

The extremal temperature problem for the rectangular lattice can be studied by using the theta functions [16, 17]. The geometry of the rectangular torus can be described uniquely by the ratio of its hottest and coldest point and this ratio refers to the elliptical modulus of the complete elliptical integral of the first kind. By using Ramanujan results, it is possible to determine the temperature on a rectangular torus lattice only if the ratios are known. This is because the elliptical modulus defines the geometry of the torus.

Now the associated heat kernel in the rectangular lattice of two extremal temperature condition [8] can be written as

$$P_\alpha\left((x_1,y_1),(x_2,y_2);t\right)=\sum_{(k,l)\in z^2} e_k^{-\lambda} t^t e_{k,l}^{\alpha}(x_1,y_1)e_{k,l}^{\alpha}(x_2,y_2)$$

For $(x_1,y_1)(x_2,y_2)\in T_\alpha^2$

Or, $$P_\alpha\left(x,y;t\right)=\sum_{(k,l)\in z^2} e^{-\pi t(\alpha^2k^2+\alpha^{-2}t^2)}e^{2\pi i(kx+ly)} \quad ...(2.19)$$

Where $t=\dfrac{1}{4\pi},\ (x,y)=\left(\alpha(x_1-x_2),\alpha^{-1}(y_1-y_2)\right)$

Then let the heat kernel $P_\alpha(x,y,t)$ associated to the torus T_α^2 can be defined in terms of maximal and minimal temperatures for the fixed time t, is given by

$$A(\alpha,t)=\min_{(x,y)} P_\alpha\left(x,y;t\right) \quad ...(2.20)$$

and $$B(\alpha,t)=\max_{(x,y)} P_\alpha\left(x,y;t\right) \quad ...(2.21)$$

respectively. From equations (2.9), the minimal and maximal temperatures in equations (2.20) and (2.21) can be represented by the hypergeometric functions [8]

$$A(\alpha,t) = P_\alpha\left(\frac{1}{2},\frac{1}{2};t\right) \quad ...(2.22)$$

and $$B(\alpha,t) = P_\alpha(0,0;t) \quad ...(2.23)$$

Now the heat kernel of equation (2.19) can be written in terms of theta function by using the relation (2.18)

$$P_\alpha(x,y;t) = \gamma_3(x,e^{-\pi t\alpha^2}).\gamma_3(y,e^{-\pi t\alpha^{-2}}); x,y \in R, t \in Z^+$$

Thus $$A(\alpha,t) = \theta_4(e^{-\pi t\alpha^2}).\theta_4(e^{-\pi t\alpha^2}) \quad ...(2.24)$$

and $$B(\alpha,t) = \theta_3(e^{-\pi t\alpha^2}).\theta_3(e^{-\pi t\alpha^2}) \quad ...(2.25)$$

If $\alpha = 1$, then from (2.24) and (2.25), the heat kernel $P(1,t)$ of the square torus of surface area $t^{-1}, t \in R^+$ and $k' \in (0,1)$, the ratio of the coldest and hottest temperature, then from (1.6), (2.9), coldest and hottest temperature on the torus $T^2(1,t)$ is given by the hypergeometric functions

$$A(1,t) = k'\ {}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ & 1\end{matrix} ;k^2\right] \quad ...(2.26)$$

and $$B(1,t) = {}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2} \\ & 1\end{matrix} ;1-k'^2\right] \quad ...(2.27)$$

respectively. This completes the proof of the theorem.

2.2.2. The interpretation of the results

The different relationship between the heat kernel and the temperature ratios between the coldest and hottest temperature expressed by the equations (2.26) and (2.27) are shown by the figures 3 and 4 respectively. The numerical value of above equations (2.26, 2.27) on RHS are obtained through Mathematica.

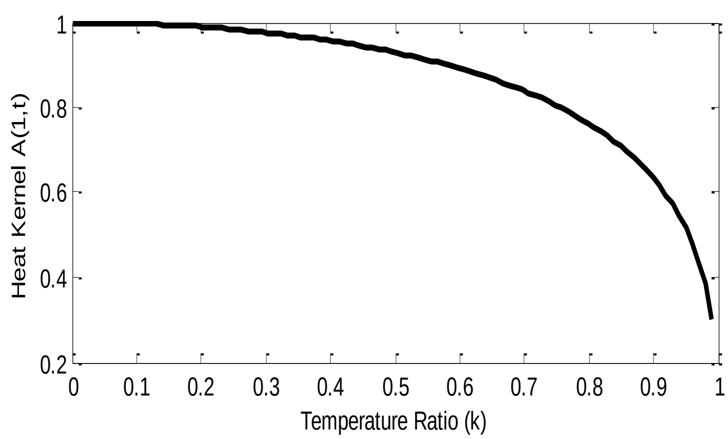

Figure 3: Relation between the temprature ratio and heat Kernel

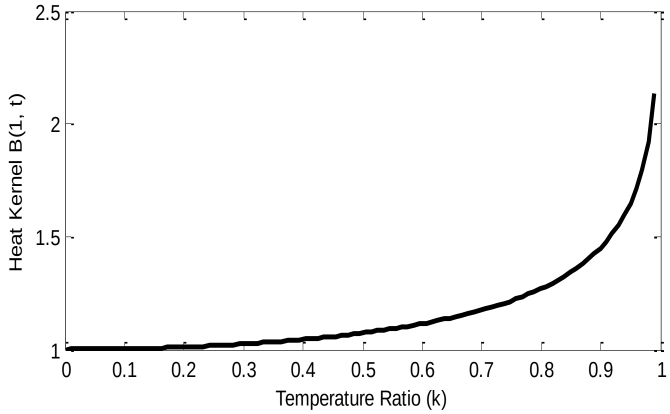

Figure 4: Relation between the conjuagate temprature ratio and heat Kernel

The figure given below (figure 5) shows the comparative behaviour of the heat kernel with respect to the temperature ratio. The temperature ratio and its conjugate are symmetrical about the line $y = 1$. The curves for $A(1,t)$ and $B(1,t)$ are asymptotic in nature and tends to infinity at $k = 1$.

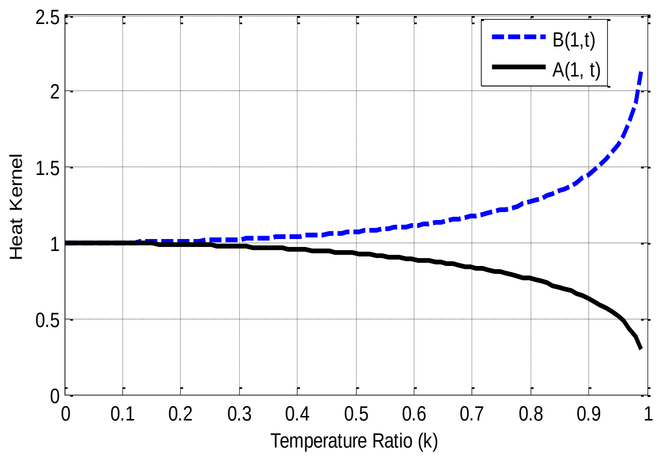

Figure 5: Comparative curve of A (1, t) and B(1, t)

The relationship between the temperature ratio (k) and its conjugate (k') are explicitly shown in the figure given below. The equation (2.11) is described by the figure 6.

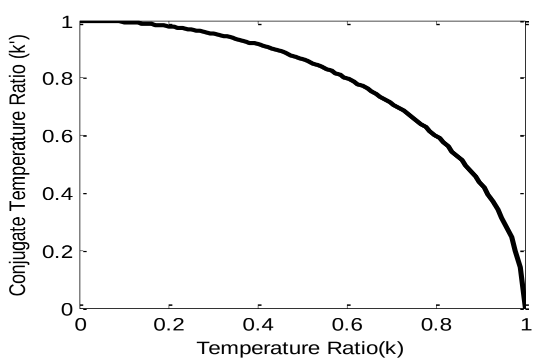

Figure 6: Relationship between temperature and conjugate Ratio

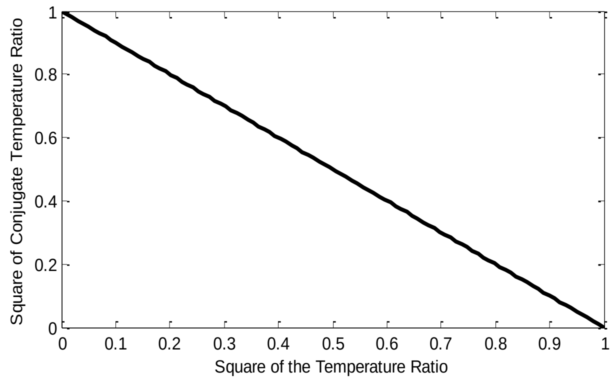

Figure 7: Relationship between squares of temperature ratio and its conjugate

3. Conclusions

In this paper, hypergeometric function is briefly introduced togather with its application in heat Kernel in a torus surface with a rectangular lattice. The guassian hypergeometric function, theta fuction, theta null and the Ramanujan Identities in [8] are used to simplify the mathematical relations involved in the heat equations. Due to the elliptical nature of the torus surface, the heat kernel relations on the temprature ratio of the coldest and hottest temprature are conjuage to each other. The various relations among the hottest and cpldest surface, eeliptical nature of the conjugate ratios are shown through the graphs on figures 3, 4, 5, 6 and 7. Most of the relations shown here are applicable in thermodynamics, rocket propulsion technology, mechanical engineering and some other branches of applied science.

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