**Far Western Review**
A Multidisciplinary, Peer Reviewed Journal
ISSN: 3021-9019
Published by Far Western University
Mahendranagar, Nepal

# University Teachers' Practices in Teaching Algebra: Students' Engagement in Constructing Multiple Representations

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## Abstract

Engaging students in constructing multiple representations and translating between them help students to become proficient in learning mathematics. The fact may also apply to abstract courses such as algebra at the university level. However, to what extent and in what ways teachers are engaging their students in constructing multiple representations while teaching algebra at the undergraduate and graduate levels, particularly in Nepal, was not explored. In this regard, the study aimed to explore university teachers' practices of applying multiple representations in facilitating algebra. It is a qualitative research under an interpretive research paradigm. I purposively selected five teachers who had been teaching algebra at the undergraduate or graduate level for at least five years. I prepared interview guidelines, including points concerning the way of teaching definitions, theorems, and proving theorems. I conducted interviews with all five participants. Then, I transcribed the interview data, coded the data, and generated four themes: Dominance of algebraic/verbal representations; variation among teachers in selection of representations; nature of content and students' academic level provide a basis for selecting representations; and graphical representation as most helpful and challenging. I interpreted themes with the theories concerned with representations in mathematics. The conclusion was that although there was variation among teachers regarding the selection of a particular type of representation, the common practice was that verbal/symbolic representations were highly used and graphical representation was rarely used by teachers, depending upon the nature of the content. Moreover, low engagement of students in making connections among four representations indicates the need of searching optimal strategy of addressing this issue to ensure conceptual understanding. The results of this study may be helpful for teachers to use different representations appropriately in teaching, students to improve their meaningful learning, and textbook writers to make books multiple representation-friendly.

***Keywords:*** Algebra, representations, students' engagement, teachers' practice, understanding

## Introduction

The nature of algebra that students study at undergraduate and graduate levels in different universities is abstract. Abstract mathematics relies on the axiomatic system (Judson & Beezer, 2020). Any axiomatic system consists of undefined terms (some terms that we do not define), definitions (precise description of the terms), axioms (statements that are accepted without proof), and theorems (statements that are derived from axioms and definitions) (Wallace & West, 1998). One of the major objectives of higher-level abstract mathematics is to enable students to prove theorems (Judson and Beezer 2020). In almost all the universities in Nepal, such as Far Western University (FWU), Mid-West University (MU), Tribhuvan University (TU), and Kathmandu University (KU), abstract algebra and linear algebra are included in the undergraduate and graduate level mathematics/mathematics education programs. In their curricula, emphasis is given to develop understanding of concepts of algebraic terms, skills of developing proofs, and solving problems associated with the theorems. Moreover, both the courses abstract algebra and linear algebra are based on axiomatic systems.

In abstract algebra, generally, students study algebraic structures such as group theory, ring theory, field theory, and Galois theory (Alam & Mohanty, 2024; Bhattacharya et al., 1995; Dummit & Foote, 2003; Fraleigh, 2003; & Hungerford, 1974). All of these are axiomatic structures. The prerequisites of abstract algebra are elementary facts about sets, functions, integers, real numbers, and mathematical maturity to understand logic (Hungerford, 1974). Similarly, linear algebra deals with vector spaces, maps between spaces, and linear maps and matrices (Lang, 1986). In different universities, except the level of complexities of the content, the above-mentioned topics are included in the undergraduate and graduate level mathematics education/mathematics programs. Looking from the educational perspective, it is good to discuss students' difficulties in learning algebra, teachers' pedagogical approaches to facilitate algebra, and ways of facilitating algebra for a good understanding of students, etc.

As already mentioned above, concept understanding and proving theorems are important aspects of teaching algebra at the undergraduate and graduate levels. By studying eight undergraduate mathematics students, six undergraduate mathematics students and two graduate mathematics students, Moore (1994) explored seven sources of difficulties that students face while proving theorems. He mentions that not knowing definitions, little intuitive understanding of concepts, inadequate concept images, lack of ability to generate examples of concepts, and lack of knowledge of applying definitions in proving are major sources associated with concept understanding. Moreover, he states that weak performance in language and notations, and a lack of the idea of sketching a proof, are sources of difficulties for developing proofs. Here, concept definitions represent formal definitions to specify that concept, and the concept image means the overall cognitive image generated in the mind of the learners (Tall & Vinner, 1981). Regarding the difficulties experienced by university-level students, Guiler (2016) conducted a study by taking semi-structured interviews of five professors, five associate professors, and five assistant professors of mathematics or mathematics education in seven different universities in Turkey. He states that lack of prior knowledge, lack of ability of understanding of proof methods used by teachers, lack of knowledge of how to start proof, negative attitude towards proof created difficulty in learning proof of theorems. Likewise, by studying five first-year graduate students, Subedi (2020) found that graduate students face difficulty in conceptualizing algebraic concepts, constructing examples and non-examples, selecting an appropriate path of proving, and understanding logical arguments. Moreover, Agustyaningrum et al. (2018) studied 31 fifth-semester students of the mathematics education program to analyze students' errors in solving abstract algebra tasks and reported that the maximum of the errors (57.94%) were conceptual errors. Here, conceptual errors mean errors due to misconceptions or a faulty understanding of the ideas and principles underlying problems, and such errors can be errors in understanding, interpreting, or applying mathematical concepts. The authors also mention that conceptual errors were due to a lack of understanding of concepts, and this lack results in poor performance in solving abstract algebra problems. Thus, in comparison to computational errors, students' errors occurred at the conceptual level.

The sources of errors explored by Moore (1994), type of errors explored by Agustyaningrum et al. (2018), and difficulties experienced by students explored by Guiler (2016) indicates that teachers need to help students in understanding concepts, generating examples, using definitions in proving theorems, and support them in sketching proofs, help in logical and language aspects and convince them about the importance of proving theorems. In this regard, the question may arise in the mind of the reader, '*how to teach abstract algebra and linear algebra*', '*what are the practices of teaching algebra*', '*how teachers are addressing problems of conceptual understanding*', etc.

Then, I searched pedagogical approaches adopted by teachers in teaching algebra at the university level. Dorier et al. (1998) suggested a pedagogical approach to teach concepts of linear algebra meaningfully. They state that first let the students work on some contextual examples before introducing formal theory, and the second step is to engage students in a mathematical activity that can be solved by them to make him/her analyze some possibilities of generalization. Here, the authors suggested a sequence-contextual examples to generalization-of facilitating concepts. Similarly, Fukawa-Connelly et al. (2016) conducted a survey on the 126 teachers who teach abstract algebra at the undergraduate or graduate level and found that the dominant mode of instruction (97 out of 126 agreed) is lecture, and out of 29 teachers who tried other pedagogies, 10 teachers generally switch back to lecturing at high rate. It means that the majority of teachers preferred to use the lecture method to teach abstract algebra. The dominance of the lecture method in teaching abstract algebra is also found by Johnson et al. (2018), who state that lecture is the predominant way of instruction. They added that 79 instructors out of 126 reported that they use the lecture method for more than 50% time duration in each class of abstract algebra. Moreover, they further found that more than one-third of the instructors who claimed that they use non-lecture methods also use the lecture method more than 25% of the time in each period. Thus, instructors use the lecture mode dominantly.

Highlighting the fostering of optimal conceptualization and comprehension of abstract algebra, Alam and Mohanty (2024) suggested a teaching approach in several steps. They state, ensure that students have sufficient knowledge of prerequisite concepts of algebra, begin with concrete and familiar examples of algebraic structures, try to motivate students by linking algebraic structures with real life contexts, encourage discussions on clear and precise definitions of terms, engage students in proving from simple to complex sequence, encourage students work in groups to solve problems in the exercise, discuss on the visual aids, diagrams and examples to illustrate concepts, as far as possible encourage exploration and foster discussion, and show practical real-life applications of abstract algebra. Thus, the authors explain a general way of teaching abstract algebra for better understanding.

The suggestions given by Dorier et al. (1998) and Alam and Mohanty (2024) seem to emphasize developing conceptual understanding. One of the major dimensions of conceptual understanding is the ability of students to create and translate among multiple representations of mathematical concepts. In this regard, the National Council of Teachers of Mathematics [NCTM] (2000) states that representations are central to the study of mathematics, and students can develop and deepen their understanding of mathematical concepts by creating, comparing, and using various representations. Further, it mentions that students can solve a range of algebra problems if they can move from one form of representation to another. Here, graphical representations (figures, diagrams, pictures, and drawings), numeric representations (ordered lists, tables), algebraic representations (symbols, formulas), and verbal representations (language) are used in terms of Mainali (2021). He states that for a better understanding of mathematics, students need to be able to translate from one type of representation to another. By knowing the role of multiple representations in understanding, I became curious to know the status of the pedagogical approach of algebra teachers with respect to students' engagement in constructing multiple representations. But, I did not find any research, particularly in Nepal, that explored the pedagogical practices of algebra teachers about students' engagement in constructing multiple representations. In this scenario, I conducted this study to explore university teachers' practices of applying multiple representations in facilitating algebra. I set the research question 'how do teachers engage their students in constructing multiple representations while teaching algebra at undergraduate and/or graduate levels in Nepali universities'.

**Theoretical Framework**

The main concern of this study is constructing multiple representations and translating among them; therefore, the idea of representation serves as the theoretical basis of the study. Analyzing the representations constructed by learners for a particular concept can be used to determine understanding of that concept (Hiebert & Carpenter, 1992). The authors mention two types of representations: internal (schema) and external (language, picture, drawings, symbols, etc.). They further state that we observe external representations to know about internal representations. Describing the importance of representation, NCTM (2000) explains that representations need to be treated as an essential component in supporting students' understanding of mathematical concepts and relations. For Goldin and Shteingold (2001), representations are signs, objects, graphs, diagrams, or combinations of these, and they might be mental processes as well. Similarly, Goldin (2003) describes representation as a configuration of signs, characters, objects, and icons that represent something else. Thus, representations are something, tangible or intangible, that represents something else.

Analyzing several literatures, Mainali (2021) concluded that there are four types of representations in mathematics: graphical, symbolic, verbal, and numeric. The graphical representations are figures, diagrams, pictures, drawings, etc. that learners make to represent mathematical concepts or mathematical situations. The algebraic representations are symbolic forms of mathematical concepts. Verbal representations are spoken languages or written language to represent mathematical concepts or situations. Numeric representations are tables, numeric lists, and ordered lists that are used to represent mathematical concepts. For example, 'the product of two real numbers multiplied in any order is same' is verbal representation, '$\forall a, b \in \mathbb{R}, a.b = b.a$' is algebraic representation, '$2 \times 3 = 3 \times 2, 6 \times 8 = 8 \times 6, etc.$' is numeric representation, and 'figures representing total number of marbles in two groups of threes and three groups of twos are same' is graphical representation of the commutative law of product. In this research, I used this theoretical framework to design interview guidelines, to conduct interviews, to code the interview data, and to analyze and interpret the data obtained from the interviews. I used four modes of representation: algebraic, graphical, verbal, and numeric as a theoretical basis of the study.

## Research Methodology

For this research, I used a qualitative approach under an interpretive paradigm (Taylor & Medina, 2011). The ontology of the research is the existence of multiple realities (Guba & Lincoln, 1994), and the epistemology is subjective. As a research tool, I prepared an interview guideline including the topics of discussion concerning the approach of teaching definitions and the approach of teaching theorem proving. The topics under discussion are concerned with the way of engaging students in constructing multiple representations and moving from one type of representation to another. I compared each point of the interview guideline with the purpose of the research to ensure that the discussion is within the track. Regarding participants, I planned to select teachers purposively who show their willingness to participate in the study and who have at least five years of experience in teaching algebra at the undergraduate or graduate level. Accordingly, I prepared a list of seven teachers, but two teachers from TU refused to participate, showing the reason of their busy schedules. Based on the interview guideline, I interviewed five teachers, two from FWU, two from MU, and one from KU of Nepal. As the data was saturated in several dimensions and one of the teacher from FWU had taught at TU at least twenty-five years, I did not try to select other teachers from TU. Therefore, these five teachers were the participants of this study. The average time of the interview was 40 minutes. During the interview, I asked the questions based on the responses of the teachers. To make the identity confidential, I used pseudonyms Teacher A, Teacher B, Teacher C, Teacher D, and Teacher E. I took the informed consent for the interview. I transcribed interviews, and I used the descriptive coding method to generate codes (Saldana, 2011). Then, I generated themes from the codes. In addition to the theoretical framework, I used the idea of conceptual understanding (Rittle-Johnson & Schneider, 2015) to interpret the themes. I ensured the credibility, authenticity, and integrity of the research.

## Results and Discussion

Since the paradigm of this research is interpretive, the interpretation of data is definitely subjective. Accordingly, I generated four themes: Dominance of algebraic/verbal representations, variation among teachers in selection of representations, nature of content and students' academic level provide a basis for selecting representations, and graphical representation as most helpful and challenging representation. Now, the time has come to give meaning to the generated themes.

**Dominance of Algebraic/Verbal Representations**

The reason behind mentioning algebraic and verbal representations simultaneously is that I find them very closely related in the teaching styles of the participants. Here, the slash '/' in algebraic/verbal is used to express both 'algebraic or verbal' as well as 'algebraic and verbal'. The algebraic and verbal representations were used side by side in teaching definitions, statements of theorems, and proving theorems. The conditions of selecting either of these two representations were different. All the teachers expressed that they use verbal representation for the purpose of discussion or explanation, and algebraic representation for the writing purpose. For example, Teacher C said, "I explain the definitions in a verbal way, and after explanation sometimes I write definitions in symbolic form. Regarding theorem, I use verbal way to explain and discuss on each step of proof but I write the proof in symbolic form". I noticed the similar voice for all the remaining teachers as well. I observed the simultaneous use of both representations in the voice of Teacher A. The teacher A said, "I write the proof on the left side in symbolic form, and I explain each step verbally before writing next step symbolically". In this way, the verbal representations and algebraic representations were used in multiple cycles of one after another. Since such discussion and explanations during proving and defining took a large part of class-time, it can be said that teachers use verbal or algebraic representations in the maximum time of the classroom discussion.

Teachers shared that with respect to time spent during teaching, they engage students in algebraic/verbal representation, numeric representation, and graphical representations in decreasing order. For example, Teacher E said

> Students prefer a symbolic form of writing, so I write proofs symbolically and explain them verbally. I rarely use diagrams in proving and defining because it is very difficult to think of graphical representations in abstract algebra. For most of the concepts, I discuss examples, but regarding theorems, I engage students in the verification of only a few of the theorems.

This is the justification of what I said above regarding the domination. Not only Teacher E, but all other participants expressed that they use diagrams rarely. Thus, I found that teachers used graphical representations for a minimum of the class time.

Although all four forms of representations (Mainali, 2021) were used by teachers in teaching algebra, in this study, algebraic representation (symbolic definition and symbolic steps of proofs) and verbal representation (explanation of definitions and proof steps) dominated other forms of representations. This result aligns with the finding of Bal (2014), who states that teachers use different types of representations, and among them, spoken language and algebraic representations are the most dominant representations. The author also mentions that shifting from one type of representation to another is a problem for teachers. In my study, I noticed that teachers did not engage students sufficiently in translating from one form of representation to another. I noticed some mismatch between the teachers' focus and students' possible expectation regarding representation. For example, Gnawali (2024) states that the undergraduate students feel proofs in algebra difficult because of complexity of mathematical notation, complexity of language used in algebra, and higher level of formalism and rigor. But, in practice, I found that the teachers highly focused on algebraic and verbal representations. Further, Subedi (2020) suggested that teachers need to engage students in constructing examples and non-examples. That is, students need to be engaged in numeric representations. His suggestions are based on his finding that graduate-level students have difficulty in constructing examples and non-examples.

Concerning the relation, I found lack of engagement in making connection between multiple representations during teaching of definitions as well as during teaching of theorem proving. This indicates the possibility of a lack of conceptual understanding because, for better conceptual understanding, students need to be able to create and translate among multiple representations (NCTM, 2000). In this regard, Agustyaningrum et al. (2018) studied 31 undergraduate students and found that the maximum of the errors (57.94%) were conceptual errors. It means that out of many aspects, conceptual understanding is the main aspect of teaching in which teacher need to support their students. Relating it to my study, I also realized that the teachers need to give sufficient opportunity to students in creating and translating from one form of representation to the other form to ensure conceptual understanding of algebra.

**Variation among Teachers in Selection of Representations**

I noticed at least two patterns (or sequences) of teachers' responses in selecting representations during the facilitation of definitions and proofs. Concerning the teaching of definitions, I observed different sequences. The first one is, 'example/non-example-diagrams (if possible)-verbal-symbolic'. In terms of representation, the associated sequence is 'numeric-graphical-verbal-algebraic'. For example, Teacher B said

> I consider some examples and non-examples of a concept. Then, I discuss diagrams, if possible. After this, I ask students to write a definition in an informal way and then in a formal way. Finally, we arrive at a symbolic form of definition. Sometimes, verbalization may be the final step if the symbolic form is not necessary.

The focus of Teacher B seems to be on the numeric representation and its connection with the verbal representation. Although the ultimate aim is developing an understanding of formal definition, Teacher B shared that he/she engage students highly in numeric representation and making connections with verbal representation, and supporting understanding through graphical representations. Regarding the teaching of making connections, a partially similar approach was shared by Teacher C. The teacher C said

> I consider one example and one non-example of the concept to be studied. I do not share that they are really examples and non-examples of that particular concept, but I engage students in a discussion of properties held by examples and non-examples. Then I informally enter into the definition and finally write it up formally. After that, I ask students to search for the link between the definition and the properties of the example. I rarely use diagrams in teaching definitions.

This indicates that Teacher C emphasizes in making connections between numeric and verbal representation of concept. Teacher C less emphasizes graphical representation and algebraic representation during the teaching of concepts. However, the activities of connecting verbal and numeric representations seem very engaging.

The second sequence is 'formal definition (in words or symbols)-diagram (if any)-example'. In terms of representation, the associated sequence is 'verbal/ algebraic-graphical-numeric'. Associated with this sequence, Teacher A said, "While teaching concepts, I start with formal definition (verbal or/and algebraic, e.g. group homomorphism). Then, I discuss with the help of diagrams and figures (if any, e.g., a figure representing a group homomorphism). Finally, I discuss numeric examples and non-examples". It indicates that emphasis is given on verbal/algebraic representation and other forms were used for supporting this representation. It further indicates that in the practice of Teacher A, verbal and algebraic representations were considered simultaneously, and students were engaged in making the link with the numeric representation. Thus, verbal/algebraic representations were kept at the center, and students were engaged to link with other forms of the concept. For most of the simple definitions, the sequences shared by Teacher D are similar to this sequence, and for almost all definitions, the sequence shared by Teacher E is similar to this sequence. The difference is that Teacher D followed the sequence of 'numeric-verbal/algebraic-graphical (if possible)' for the complex type of definitions, and Teacher E rarely used the graphical representations. Except for these differences, these teachers follow the above-mentioned sequence of representations. Moreover, Teacher C and Teacher D shared that for complex types of concepts (algebraic structures) they start the teaching from the real life example as far as possible. Thus, there are some deviations among teachers regarding the selection and preference of different types of representations during the facilitation of algebraic concepts.

Now, the time has come to discuss the variation among teachers regarding the representations during teaching of theorem proving. While teaching the statement of the theorem, all the teachers shared that they explain the statement verbally. Then, two different practices were noticed. Some teachers consider graphical representation by making diagrams of statements (if possible) and try to make the meaning clearer by linking the diagram and the statement of the theorem. For example, Teacher A said, "I write the statement on the left side and draw a diagram on the right side (if possible). Then I discuss the statement referring to the diagram". Similarly, Teacher B also expressed a similar approach of considering diagrams for teaching statements. The second approach was that teachers did not try to make figures for the statements, but they discuss the statements verbally. Teacher C, Teacher D, and Teacher E followed this second approach. In this regard, Teacher E said, "I write statement formally. I ask the key terms of the statement. We discuss on what is given and what is to be proved. Then we start proving". Regarding the verification, some teachers said they verify statements before proof, and some said they verify after proof. Thus, two sequences for statements are 'verbal (algebraic optional)-graphical-numeric' and 'verbal (algebraic optional)-numeric'. The data shows that students did not get sufficient opportunity to translate among multiple representations while teaching statements of theorems.

While proving theorems, all teachers shared that they discuss the sketch of proof and every step of proof verbally, and they write up the proof symbolically". In proving as well, some differences were noticed regarding verification. Teacher A, Teacher B, and Teacher E shared that they apply the theorem to verify in an example after proving the theorem. However, Teacher C shared that he/she first consider example and then discuss proof by linking the proof steps with the example. In this regard Teacher C said "I write statement formally. Then, I engage students in numerical verification of the proof. Then we discuss the possible method of proving the theorem. Finally, we discuss proof verbally and write up proof symbolically. Teacher D shared that "It is not necessary to verify every theorem. Verification of the theorem might be before or after proving, or it may not be needed". During proving, Teacher A shared that he/she use diagrams (if possible) and discuss every step of the proof by making connections with the diagram. Thus, there is variation among teachers regarding the selection of graphical representation and the order of verification. From the above data, the following three sequences can be observed: 'verbal and algebraic side by side-numeric', 'numeric-verbal and algebraic side by side', and 'verbal, algebraic, and graphical simultaneously-numeric'. Thus, the connection between algebraic and verbal representations seems strong during proving, the relationship between verbal/algebraic and numeric seems moderately addressed, but the graphical representation seems almost ignored by teachers, except Teacher A, while teaching proving theorems.

Some teachers, in this study, said that they start with examples and then discuss the formal definition. This aligns with the suggestions given by Dorier et al. (1998), who state that for meaningful teaching of algebra, the first step is to work on some contextual examples. It also aligns with the suggestions given by Alam and Mohanty (2024), who said that before entering into the formal definitions, there should be a discussion on prerequisite knowledge, real-life context, and concrete and familiar examples. However, some teachers said that they start with a formal definition and then discuss examples and non-examples, making appropriate connections. This approach satisfies some properties of the expository/lecture method. I do not say that the lecture method is a less effective method because effectiveness is reflected from the students' performance, and I did not measure students' performance in this study. I am simply reporting the two different approaches that I noticed from teachers' responses.

Further, the method of teaching of all the teachers of this study concerning theorem proving can be termed as the expository method. The expository method is common in higher-level mathematics. For example, by studying 126 algebra teachers, Fukawa-Connelly et al. (2016) found that the majority of the teachers use the lecture method. Similarly, Johnson et al. (2018) state that the majority of teachers use the lecture method more than 50% of the time in a class, and other teachers use the lecture mode of instruction more than 25% of class time. It indicates that the lecture mode of instruction is still common to different universities within and beyond Nepal. However, to ensure conceptual understanding (Rittle-Johnson & Schneider, 2015), teachers need to engage students in creating and connecting all four representations. Moreover, Alam and Mohanty (2024) mention that algebra deals with abstract concepts and structures; therefore, teachers need to follow structured inquiry-based teaching methodologies.

**Graphical Representation as Most Helpful and Challenging**

Teachers shared that out of the four representations, the graphical representations are more helpful for developing an understanding of algebraic concepts. All the teachers agree on the fact that it is either impossible or difficult to generate graphical representations of abstract algebra concepts. In this regard, Teacher A said

> When definitions are taught using diagrams, students understand best. For example, students understand the complex idea of a symmetric group easily when I taught them using diagrams. Similarly, the use of diagrams helps to understand proofs as well. For example, when I taught fundamental theorem of group homomorphism, students understood from the diagram. But generating diagrams for every definition and proof is a very difficult task.

Similarly, Teacher C said, "If we can construct an appropriate diagram, students understand easily in comparison to symbolic, verbal, or numeric forms, but in the case of abstract algebra, it can rarely be constructed". Likewise, Teacher B said "In undergraduate level it is necessary to discuss on diagrams representing definitions and statement of theorems for better understanding but for higher level concepts that we deal in masters' degree it is difficult to construct diagrams". Furthermore, Teacher D and Teacher E had common voice that "Although the diagrams and figures are very helpful for understanding underlying concepts I do rarely use diagrams because it is not easy to construct appropriate graphical representation for abstract algebra". Thus, all the teachers agree that the graphical representations are more helpful for meaningful learning, but such representations can rarely be constructed in algebra. Further, I found that teachers had been using the graphical representations to make the underlying concept meaningful.

Regarding the role of representations, NCTM (2000) states that to deepen understanding of mathematical concepts, students need to have the opportunity to create, compare, and use various representations. But teachers in this study seem they rarely used graphical representations which ultimately may lead to the lack of understanding of algebraic concepts and principles. Moreover, if the students did not get the opportunity to engage in graphical representation, then they could not see the algebra visually, which ultimately might develop the attitude that algebra is really abstract, formal, and has a rigorous nature. This attitude hampers students in learning algebra (Gnawali, 2024). Highlighting the role of visuals in algebra, Unal et al. (2023) studied sixty ninth-grade students in Turkey and found that algebraic visuals are highly associated with the enhancement of algebraic reasoning. It means that students can develop algebraic reasoning if they have the opportunity to deal with algebraic visuals. But in this study, these visual aspects seem weak in the practices of teachers. Therefore, it seems that the teachers need to use diagrams and figures while teaching algebraic concepts and theorems.

**Nature of Content and Students' Level Provide Basis for Selecting Representations**

I noticed in the responses of the teachers that the nature of the content to be taught and the students' academic level play a vital role in the selection of the type of representation. Regarding the nature of the content, Teacher D said

> I do not follow same sequence for all concepts. If the definition of a concept is easy, then I follow a deductive approach. It means first I explain the formal definition and then I ask students to verify examples and non-examples. If the definition is hard to understand, I follow an inductive approach. It means, first we discuss the example and then I explain the formal definition by linking it with the example. Thus, sequences 'verbal-numeric' or 'numeric-verbal' depend upon the

> complexity of the definitions. Moreover, while teaching theorem proving, if the proof is simple, I first prove the theorem and then I ask students to verify it. If the proof is complex, first I consider a numeric example and then I teach formal proof.

Thus, for teacher D, preference for a particular representation depended upon the nature of the complexity of the definitions or theorems. Similarly, Teacher C said, "I rarely use the diagrams in abstract algebra, but while teaching linear algebra, I prefer to use graphical representation. Moreover, whenever the content is related to functions or sets, I try to consider a graphical representation". This also indicates that the selection of representation depends upon the nature of the content. Furthermore, all the teachers had been using verbal representation for explaining proof, algebraic representation for writing proof, and verbal representation for the statement of a theorem, which also indicates that the nature of the content provides a basis for the selection of representation.

Similarly, I noticed that the academic level of students also plays a role in the selection of representation. Concerning this, Teacher B said, "In the undergraduate level, I give emphasis on numerical and graphical representations where as in the graduate level I give emphasis on verbal representation". Similarly, Teacher A and Teacher D had the same voice that "In comparison to graduate level I explain definitions and theorems more in undergraduate level". Thus, in the undergraduate level, teachers give more place to numeric and graphical representations in comparison to the graduate level. Further, for similar contents at graduate and undergraduate levels, demanding verbal representation, comparatively high engagement of teachers is found at the undergraduate level. Thus, depending upon the academic level, teachers select different representations for the similar content.

For a better understanding of any mathematical concept, students need to create and move from one representation to another (Mainali, 2021). However, if the content to be learned is very simple, teacher might not give time for creating all four types of representations and if the content is very difficult teacher might give more time for numeric representation to support verbal and symbolic representation like teaching approach shared by Teacher D. Further, concepts in abstract algebra are abstract in nature (Alam & Mohanty, 2024) so teachers and students might feel difficult to create graphical representations in abstract algebra, which is similar to the finding of this study. In addition, mathematical maturity is one of the prerequisites for learning algebra (Hungerford, 1974), and this maturity is definitely better at the graduate level in comparison to the undergraduate level, therefore, the variation on selections with respect to the level of students in this study seems reasonable. Therefore, keeping students' intellectual level and their mathematical maturity in the mind, the participant teachers might have followed different representational preferences in the graduate and undergraduate levels.

## Conclusion

Dominance of algebraic and verbal representations during facilitation of concepts and proofs of algebra at undergraduate and graduate levels shows that students are getting little opportunity to create and move from one type of representation to another. Therefore, there is space for making students' understanding better by engaging students in creating numeric and graphical representations and making connections among all four representations. Teachers have their beliefs regarding students' learning, and they possess different teaching styles. Accordingly, they may prefer different representations for the same content, and they may follow a different sequence of multiple representations during algebra teaching. Which sequence is better, and which approach is better for understanding algebra, is not explored in this research. This may be the topic for further research. From the acceptance of graphical representation as the most helpful representation for understanding algebra, it can be inferred that teachers need to give sufficient time to discuss the graphical representation of algebraic concepts. In addition, acceptance of the work of creating graphical representations as a challenging task and the situation of rare use of graphical representations opens the door for doing research on developing graphical representations for all the algebraic concepts, whenever possible. As the preference and sequence of selection of different modes of representations largely depend upon the nature of the content and students' academic and intellectual level, teachers need to be careful about the appropriate selection of representations. The results of this study may help algebra teachers to enhance students' conceptual understanding by careful consideration of different modes of representation. The textbook writers may use the results of this study to adjust content in such a way that it gives enough opportunity for moving from one mode of representation to another. The students at the university level may use the results of this study to enhance their learning style of algebra.

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