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Computational Methods of Ruin Probability: Actuarial Comparison of De-Vylder and Tijim's Models

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Abstract

The underwriting operation of insurance firms is to assume the risk of the insured in return of premium received. In order to shield itself against extreme losses and avoid the risk of insolvency, it becomes necessary to examine how the portfolio is expected to perform over a long time horizon. The surplus process is connected with the excess of the premium received over claims outgo of the insurer's portfolio to enable it predict the level at which the insurer could survive. When the surplus approaches a defined lower limit irrespective of the initial reserve, then the insurer is ruined. The probability of ruin apparently defines the volatility embedded in underwriting process as a useful tool of risk measurement in long range planning of premium rate. This paper is anchored on the following objectives: solve the adjustment co-efficient using the moment generating function, compute the De-Vylder's ruin, compute the Tijim's ruin approximation and then compare the two. Although it was verified that as the initial capital increases, the probability of ruin decreases under the two models, computational evidence from our results in tables 2 and 3 shows that the Tijim's approximation to ruin probability is higher than the De-Vylder's approximation at the same level of initial capital and safety loading. The implication is that the De-Vylder's approximation to ruin probabilities is an improvement over Tijim's ruin model and hence De-Vylder's approximation is recommended for the insurer's ruin assessment.

Keywords: Adjustment coefficient, minimum capital, ruin probability, surplus process, underwriting operations

Introduction

When investigating the degreeof the risk connected with a portfolio of insurance policies, it becomes necessary to examine the expected performance of the portfolio over a long time horizons. In anticipation of the potential severity, the insurer usually sinks a reserve amount from which fund could be drawn as claim arises. In practice, the reserve is accumulated over a long time horizon either from the excess of premiums received or from the investment income. The surplus process is a useful analytical framework for examining how an insurer's capital evolve over time horizon. Consequently, ruin theory is connected with the excess of the premiums received over claims paid in modelling the surplus process. The surplus process $Z(\xi)$ commences with an initial capital u and it is modelled based onboth premium income per unit time and aggregate claim amount which are respectively defined in terms of cash in-flows and the cash out-flows up to time and this accounts for the reason why the classical risk process is defined as a function of the initial surplus, premiums and claims. Ruin therefore arises whenever the insurer's surplus approaches a specified lower bound net of the initial reserve.

Centeno (1986) derived the insurer's adjustment coefficient as a function of the retention levels under the assumption that the annual claim is distributed as compound Poisson. Liang and Guo (2007) obtained the insurer's surplus process using the Brownian motion where closed form expression was derived in the diffusion approximation conditions to obtain an optimal strategy in minimizing the ruin probability by maximizing the adjustment coefficient. However, under the invariance hypothesis of Brownian motion Korzeniowski (2023) obtained a closed form expression of ruin probability under the discrete time risk model with random premiums.

Guerra and Centeno (2008) examined the adjustment coefficient of an arbitrary optimal reinsurance schemes on the basis of expected utility. The author further obtained a functional relationship between maximizing the adjustment coefficient and maximizing the expected utility of wealth. Burnecki, Teuerle, Wilkowska (2019) experimented the De-Vylder estimations of the ruin probability for a two-dimensional surplus process where claims and premiums are divided under a predetermined percentage. Burnecki, Mista and Weron (2003) developed a generalized De-Vylder approximation. The rationale behind this technique is to develop an advanced model for the ruin probability based on gamma claims and matching first four moments.

Santana & Rincon (2023) obtained a new model for the ultimate ruin probability in the Cramer–Lundberg risk process where the claims assume a finite mixture of k Erlang distributions. Employing the power of recurrence sequences, the technique suggested here resulted in computing ruin probability using an associated characteristic polynomial and its roots. The model is obtained through a finite sum of terms one for each zero of the polynomial in order to approximate the ruin probability.

Michna (2020) derived a model for the supremum distribution of a specified positive or negative Levy processes characterized by a broken linear drift. This resulted in an alternative model for ruin probabilities given that two underwriting firms firms divide between them both claims and premiums in some prescribed proportions

Luesamai (2021) derived a lower bound to obtain the finite time ruin probability that converges to the ultimate ruin probability with increasing time while the upper bound is iteratively estimated as initial point. Huang, Li, Liu and Yu (2021) estimated the ruin probability in insurance risk model under stochastic premium income where the ruin probability was computed through complex Fourier series expansion methodology.

According to Cheng, Gao and Wang (2016); Karageyik and Sahin (2016), Ogungbenle (2024), the surplus expressed in terms of the insurer's risk process $\left(U(\xi)\right)_{\xi\geq 0}$ is expressed as

$$U(\xi) = u + \pi\xi - Z(\xi) \quad (1)$$

u is the initial capital, π is the premium per unit time paid by the insured and $Z(\xi)$ is the aggregate claim advised against the insurer.

(2)

$$Z(\xi)=\sum_{i=1}^{m(\xi)} X_i$$

X_i is the mth claim and $M(\xi)$ is a poisson process. Consequently, we can obtain then obtain the conditional expectation of the total claim size as follows in equation (3) where $\mathbf{E}$ and $\mathbf{P}$ are expectation operator and probability function. Observe that

$$\mathbf{E}\left(Z(\xi)\right)=\mathbf{E}\left[Z(\xi)\middle|m(\xi)=m\right]=m\mu \tag{3}$$

$$\mathbf{E}\left(Z(\xi)\right)=\sum_{m=0}^{\infty}\left(\left(\mathbf{E}\left[Z(\xi)\middle|m(\xi)=m\right]\right)\times\mathbf{P}\left(m(\xi)=m\right)\right) \tag{3a}$$

Putting equation (3) in (3a), we have

$$\mathbf{E}\left(Z(\xi)\right)=\sum_{m=0}^{\infty}\left(m\mu\times\mathbf{P}\left(m(\xi)=m\right)\right) \tag{4}$$

Observe that the probability mass function of $m(\xi)$ is Poisson and hence we have

$$\mathbf{P}\left(m(\xi)=m\right)=\frac{(\lambda\xi)^m e^{-\lambda\xi}}{m!} \tag{5}$$

Applying equation (4) we obtain the expectation (the first moment) of the total claim size as follows

$$\mathbf{E}\left(Z(\xi)\right)=\sum_{m=0}^{\infty}\left(m\mu\times\frac{(\lambda\xi)^m e^{-\lambda\xi}}{m!}\right) \tag{6}$$

$$\mathbf{E}\left(Z(\xi)\right)=\mu(\lambda\xi)e^{-\lambda\xi}\sum_{m=1}^{\infty}\left(\frac{(\lambda\xi)^{m-1}}{(m-1)!}\right) \tag{7}$$

Let $Y=m-1$ in equation (7), then we have

$$\mathbf{E}\left(Z(\xi)\right)=\mu(\lambda\xi)e^{-\lambda\xi}\sum_{m=1}^{\infty}\left(\frac{(\lambda\xi)^{Y}}{Y!}\right) \tag{8}$$

But observe that by definition

$$e^u=\sum_{m=0}^{\infty}\frac{u^m}{m!} \tag{9}$$

Consequently, equation (8) becomes

$$\mathbf{E}\left[Z(\xi)\right]=\mu e^{-\lambda\xi}\times\lambda\xi e^{\lambda\xi} \tag{10}$$

Therefore,

$$\mathbf{E}\left[Z(\xi)\right]=\mu\lambda\xi \tag{11}$$

We can now obtain the second moment as follows

$$\mathbf{E}\left[Z^2(\xi)\right]=\mathbf{E}\left[\left(\sum_{m=1}^{m(\xi)}X_k\right)^2\right] \tag{12}$$

By the law of total expectation

$$\mathbf{E}\left[Z^2(\xi)\right]=\sum_{m=0}^{\infty}\left(\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]\times\mathbf{P}(m(\xi)=m)\right) \tag{13}$$

Observe that $Z^2(\xi)$ is the squared aggregate of m independently and identically distributed random claims

$$Z^2(\xi)=\left(X_1+X_2+X_3+\ldots+X_m\right)^2 \tag{14}$$

$$Z^2(\xi)=X_1^2+X_2^2+X_3^2+\ldots+X_m^2+\ldots+2X_1X_2+2X_1X_3+\ldots \tag{15}$$

Therefore

$$\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]=\mathbf{E}\left(X_1^2\right)+\mathbf{E}\left(X_2^2\right)+\mathbf{E}\left(X_3^2\right)+\ldots+\mathbf{E}\left(2X_1X_2\right)+\mathbf{E}\left(2X_1X_3\right)+\ldots \tag{15}$$

Since the random losses are independently and identically distributed, we have

$$\mathbf{E}\left(2X_1X_2\right)=2\mathbf{E}\left(X_1\right)\mathbf{E}\left(X_2\right) \tag{15a}$$

and

$$\mathbf{E}(X)=\mathbf{E}(X_1)=\mathbf{E}(X_2)=\mathbf{E}(X_3)=...=\mathbf{E}(X_m) \tag{16}$$

Therefore,

$$\mathbf{E}(X_1X_2)=\mathbf{E}(X_2X_3)=\mathbf{E}(X_1^2)=\mathbf{E}(X_2^2)=...=\mathbf{E}(X_m^2)=\mathbf{E}(X^2) \tag{17}$$

$$\mathbf{E}(X_1X_2)=\mathbf{E}(X_1)\mathbf{E}(X_2)=\mu^2 \tag{18}$$

$$\mathbf{E}(X^2)=\mathbf{E}(X_1^2)=\mathbf{E}(X_2^2)=\mathbf{E}(X_3^2)=...=\mathbf{E}(X_m^2) \tag{19}$$

$$\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]=m\mathbf{E}(X^2)+m(m-1)(\mathbf{E}(X))^2 \tag{20}$$

$$\mu_2=\mathbf{E}(X^2) \tag{21}$$

$$\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]=m\mu_2+m(m-1)\mu^2 \tag{22}$$

$$\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]=m\mu_2+m(m-1)\mu^2 \tag{23}$$

$$\mu_2=\sigma^2+\mu^2 \tag{24}$$

$$\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]=m(\sigma^2+\mu^2)+m(m-1)\mu^2 \tag{25}$$

$$\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]=m\sigma^2+m\mu^2+m^2\mu^2-m\mu^2 \tag{26}$$

$$\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]=m\sigma^2+m^2\mu^2 \tag{27}$$

$$\mathbf{E}\left[Z^2(\xi)\right]=\left\{\sum_{m=0}^{\infty}\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]\right\}\times f_m(\xi) \tag{28}$$

$$\mathbf{E}\left[Z^2(\xi)\right]=\left\{\sum_{m=0}^{\infty}\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]\right\}\times \mathbf{P}(m(\xi)=m) \tag{29}$$

Equation (4) is a density function and can be rewritten as

$$f_m(\xi)=\frac{(\lambda\xi)^m e^{-\lambda\xi}}{m!} \tag{30}$$

Then at the extended point $(\xi+s)$, we have

$$f_m(\xi+s)=\sum_{r=0}^{m}f_r(\xi)f_{m-r}(s),\ m=0,1,2,... \tag{31}$$

But

$$\sum_{m=0}^{\infty}\mathbf{E}\left[Z^2(\xi)\middle|m(\xi)=m\right]=\left\{m\sigma^2+m^2\mu^2\right\} \tag{31a}$$

Substituting (31a) in (29), we have

$$\mathbf{E}\left[Z^2(\xi)\right]=\sum_{m=0}^{\infty}\left\{\left\{m\sigma^2+m^2\mu^2\right\}\times\frac{(\lambda\xi)^m e^{-\lambda\xi}}{m!}\right\} \tag{32}$$

Simplifying equation (32) further, we have

$$\mathbf{E}\left[Z^2(\xi)\right]=\sum_{m=0}^{\infty}\left(m\sigma^2\frac{(\lambda\xi)^m e^{-\lambda\xi}}{m!}\right)+\sum_{m=0}^{\infty}\left(m^2\mu^2\frac{(\lambda\xi)^m e^{-\lambda\xi}}{m!}\right) \tag{33}$$

$$\mathbf{E}\left[Z^2(\xi)\right]=\sum_{m=0}^{\infty}\left(\sigma^2\frac{(\lambda\xi)^m e^{-\lambda\xi}}{(m-1)!}\right)+\sum_{m=0}^{\infty}\left(m\mu^2\frac{(\lambda\xi)^m e^{-\lambda\xi}}{(m-1)!}\right) \tag{34}$$

Now the first two moments of the counting process $\mathbf{E}(m(\xi))$ are given in equations (35) and (35a) to enable us obtain the first two moments of the total claim size

$$\mathbf{E}(m(\xi))=\sum_{m=0}^{\infty}\left(\frac{(\lambda\xi)^m e^{-\lambda\xi}}{(m-1)!}\right) \tag{35}$$

$$\mathbf{E}\left[m^2(\xi)\right]=\sum_{m=0}^{\infty}\left(m\mu^2\frac{(\lambda\xi)^m e^{-\lambda\xi}}{(m-1)!}\right) \tag{35a}$$

$$\mathbf{E}\left[Z^2(\xi)\right]=\sigma^2\mathbf{E}(m(\xi))+\mu^2\mathbf{E}(m^2(\xi)) \tag{36}$$

$$\mathbf{E}\left[Z^2(\xi)\right]=\sigma^2\mathbf{E}\left(m(\xi)\right)+\mu^2\left\{VAR\left(m(\xi)\right)+\left[\mathbf{E}\left(m(\xi)\right)\right]^2\right\} \quad (37)$$

Again, observe that

$$\left(m(\xi)\right)=VAR\left(m(\xi)\right)=\lambda\xi \quad (38)$$

$$\mathbf{E}\left[Z^2(\xi)\right]=\sigma^2\lambda\xi+\mu^2\left\{\lambda\xi+\lambda^2\xi^2\right\} \quad (39)$$

$$\mathbf{E}\left[Z^2(\xi)\right]=\sigma^2\lambda\xi+\lambda\xi\mu^2+\lambda^2\xi^2\mu^2=\lambda\xi\left(\sigma^2+\mu^2+\lambda\xi\mu^2\right) \quad (40)$$

$$\mathbf{E}\left[Z^2(\xi)\right]=\sigma^2\lambda\xi+\lambda\xi\mu^2+\lambda^2\xi^2\mu^2=\lambda\xi\left(\mu_2+\lambda\xi\mu^2\right) \quad (41)$$

So that the expected value of the whole of the surplus process in equation (1) becomes

$$\mathbf{E}\left(U(\xi)\right)=u+\pi\xi-\mathbf{E}\left(Z(\xi)\right) \quad `(42)$$

$$\mathbf{E}\left(U(\xi)\right)=u+\pi\xi-\mu\lambda\xi \quad (43)$$

The probability of the ultimate ruin in continuous time for infinite time is expressed as $\psi(u)=\mathbf{P}\left(u(\xi)<0\, for\ some\ \xi>0\right)$ and representsthe probability that the insurer's surplus declines below a prescribed time in the long run and such that the claims size outgo exceed the initial surplus plus premium income.In the classical risk process, $m(\xi)$ is a homogeneous Poisson process with intensity λ while $U(\xi)$ represents the classical risk model or Cramer-Lundberg model. The Cramer's condition is obtained by applying theEsscher transform

$$\int_0^\infty e^{Rx}\left(S_X(x)\right)dx=\mu(1+\theta) \quad (44)$$

Observe that $e^{Rx}\left(S_X(x)\right)\to 0$ as $x\to\infty$ and $S_X(x)\to 0$ faster than e^{-Rx} making the tail of $1-F_X(x)$ to become light. We observe a counter example for the Pareto density

$$f_X(x)=\alpha\frac{\beta^\alpha}{x^{\alpha+1}} \quad (44a)$$

The tail of Pareto distribution is

$$1-F_X(x)=\left(\frac{\beta}{x}\right)^\alpha \quad (44b)$$

Therefore, $e^{Rx}\left(1-F_X(x)\right)=e^{Rx}\left(\frac{\beta}{x}\right)^\alpha\to\infty$ for $R>0$ and hence the Lundberg exponent does not exist and consequently the above counter example implies that ruin probability can violate the Lundberg's upper bound particularly for large initial capital.In a risk process where claim sizeare $X\geq 0$ and $\mathbf{E}(X)=\mu>0$, the adjustmentequation has a trivial solution but the adjustment coefficient R is a positive real number satisfying the equation

$$M_X(R)-1=(1+\theta)\mu(R) \quad (45a)$$

The adjustment coefficient is then solved from

$$\lambda+\pi R=\lambda M_X(R) \quad (45c)$$

Suppose Z is the total claimsizein an interval of length 1, then $(\pi-Z)$ defines the profit in that interval. We assume further that Z is a compound Poisson with intensity λ and the moment generating function of Z is

$$M_{Z_\xi}(R)=M_{m_\xi}\left(\log_e M_X(R)\right)=e^{\lambda\xi\left(M_X(R)-1\right)} \quad (45b)$$

Differentiating equation (45b) once, we obtain

$$M_{Z_\xi}'(R)=\lambda\xi M_X'(R)e^{\lambda\xi\left(M_X(R)-1\right)} \quad (45c)$$

From (45c),

$$M_{X'}(0)=1 \quad (45d)$$

$$M_{Z_\xi}'(0)=\lambda\xi M_X'(0)=\lambda\xi\mu \quad (46)$$

The equation (46) could be computationally prohibitive to solve butit can be employed to derive an inequality. For compound Poisson process, the upper bound for the infinite time ruin probability yields that

$$\psi(u)\leq e^{-Ru} \quad (47)$$

The Cramer-Lundberg's asymptotic infinite time ruin probability model for large value of initial capital u is given as $\psi(u) \approx Be^{Ru}$ where

$$B = \frac{\theta\mu}{M'_X(R) - \mu(1+\theta)} \quad (48)$$

The ruin probability decreases quickly when the adjustment coefficient increases.The Cramer-Lundberg's model requires that the adjustment coefficient exists and assumes light-tailed distributions. However if the individual claim value isexponentially distributed with the parameter α , the precise infinite time ruin probability will be obtained as $\psi(u) = \psi(0)e^{-Ru}$ where $\psi(0) = \frac{1}{(1+\theta)}$ and $R = \alpha - \frac{\lambda}{\pi}$

The Adjustment Coefficient

Suppose $\mathbf{P}$ defines a probability measure with

$$\mathbf{E}(\mathbf{P}) = \int x d\mathbf{P}(x) \in]-\infty, 0[\quad (48a)$$

The moment generating function defined by $M_P(\beta) = \int e^{\beta x} d\mathbf{P}(x)$ for finite $M_P(\beta) < \infty$ for some $\alpha > 0$ and $\lim_{\beta \to \xi^-} M_P(\beta) \geq 1$ for $\xi = \sup\{\beta > 0 : M_P(\beta) < \infty\}$. If$\exists$ a real number $R > 0$ such that $M_P(R) = 1$, then R is the adjustment coefficient of $\mathbf{P}$. The adjustment coefficient R exists under the following conditions

$M_P(0) = 1, M^{(1)}{}_P(0) < 0$ and $M_P(\beta)$ is strictly convex as $M^{(2)}{}_P(\beta) > 0$ for all $\beta < \xi$. The number R is the only positive number such that $e^{Rx} dP(x)$ is a probability measure.

Lundberg Inequality

Let $(Z_n : n \in \mathbf{N})$ define a sequence of independent and identical random losses with distribution P

Let

$$\Psi(u) = \mathbf{P}\left(\sup\left\{ \sum_{t=1}^{m} Z_i : m \in \mathbf{N} \right\} > u \right) \quad (49)$$

for all $u \geq 0$

Define

$$\Psi_n(u) = \begin{cases} \mathbf{P}\left(\sup\left\{ \sum_{i=1}^{m} Z_i : m \leq n \right\} > u \right), & u \geq 0 \\ 1, & u < 0 \end{cases} \quad (50)$$

For all $n \in \mathbf{N}$, we show that $\Psi_n(u) \leq e^{-Ru}$ for all $u \in \mathbf{R}$ by inductive principles on n. Observe that

$$\Psi(u) = \lim_{n \to \infty} \Psi_n(u)(u) \quad (51)$$

For $u \leq 0$ and all $n \in \mathbf{N}$, the bound is true from definition. Let $n = 1$ and $u > 0$. Then

$$\Psi_1(u) = \mathrm{P}(Z_1 > u) = \mathrm{P}(RZ_1 > Ru) \quad (52)$$

$$\Psi_1(u) \leq e^{-Ru}\mathbf{E}\left(e^{RZ_1}\right) = e^{-Ru} \quad (53)$$

By the Independence assumption of Z_n, we have

$$\Psi_{n+1}(u) = 1 - \int_{-\infty}^{u} 1 - \Psi_n(u-z) d\mathbf{P}(z) \quad (54)$$

$$\Psi_{n+1}(u) = 1 - P(]-\infty, u]) + \int_{-\infty}^{u} \Psi_n(u-z) d\mathbf{P}(z) \quad (55)$$

$$\Psi_{n+1}(u) = \int_{-\infty}^{\infty} \Psi_n(u-z) d\mathbf{P}(z) \quad (56)$$

$$\Psi_{n+1}(u) \leq \int_{-\infty}^{\infty} e^{-R(u-z)} dP(z) = e^{-Ru} \quad (57)$$

Theorem

$$\text{If } \frac{\pi}{\alpha\mu} \leq \frac{\left(e^{RK} - 1\right)}{RK} \quad (57a)$$

then

$$\pi e^{-RK} \le \pi - \frac{\pi}{2}RK + \frac{\pi}{12}R^2K^2 - \frac{\pi}{720}R^4K^4 < \alpha\mu \quad (58)$$

Proof

$$\frac{Y}{K}e^{RK} + \left(1-\frac{Y}{K}\right) = 1 - \frac{Y}{K} + \frac{Y}{K} + YR + \frac{YR^2K}{2!} + \frac{YR^3K^2}{3!} + \frac{YR^4K^3}{4!} + \ldots \quad (59)$$

By Jensen's inequality,

$$E\left(f\left(Y\right)\right) \le f\left(\mathbf{E}\left(Y\right)\right) \quad (60)$$

$$\alpha + \pi R \le \alpha\mathbf{E}\left(e^{RY}\right) \quad (61)$$

$$\alpha + \pi R \le \alpha\mathbf{E}\left(\frac{Y}{K}e^{RK} + \left(1-\frac{Y}{K}\right)\right) \quad (62)$$

$$\alpha + \pi R \le \alpha\mathbf{E}\left(\frac{Y}{K}e^{RK}\right) + \alpha\mathbf{E}\left(1-\frac{Y}{K}\right) \quad (63)$$

$$\alpha + \pi R \le \alpha\mathbf{E}\left(\frac{Y}{K}\right)e^{RK} + \alpha\left(1-\frac{\mathbf{E}\left(Y\right)}{K}\right) \quad (64)$$

$$\alpha + \pi R \le \alpha\left(\frac{\mu}{K}\right)e^{RK} + \alpha\left(1-\frac{\mu}{K}\right) \quad (65)$$

$$\alpha + \pi R \le \frac{\alpha\mu}{K}e^{RK} + \alpha - \frac{\alpha\mu}{K} \quad (66)$$

$$\pi R \le \alpha\frac{\mu}{K}\left(e^{RK}-1\right) \quad (67)$$

$$\frac{\left(e^{RK}-1\right)}{KR} \ge \frac{\pi}{\alpha\mu} \quad (68)$$

Observe that the following inequality holds

$$\frac{\left(e^{RK}-1\right)}{RK} < e^{RK} \quad (69)$$

Implying

$$\frac{\pi}{\alpha\mu} \le \frac{\left(e^{RK}-1\right)}{RK} < e^{RK} \quad (70)$$

Taking reciprocals of all sides of the inequalities, we have

$$\frac{\alpha\mu}{\pi} \ge \frac{RK}{\left(e^{RK}-1\right)} > e^{-RK} \quad (71)$$

Applying the Euler-Maclaurin series formula (71a)

$$\frac{y}{e^y-1} = \sum_{j=0}^{\infty}\frac{B_j x^j}{j!} \quad (71a)$$

where B_j are the rational Bernoulli numbers

$$\frac{y}{e^y-1} \cong 1 - \frac{y}{2} + \frac{y^2}{12} - \frac{y^4}{720} + \frac{y^6}{30240} - \frac{y^8}{1209600} + \frac{y^{10}}{47900160} \quad (71b)$$

we have

$$\frac{\alpha\mu}{\pi} \ge 1 - \frac{1}{2}RK + \frac{1}{12}R^2K^2 - \frac{1}{720}R^4K^4 > e^{-RK} \quad (72)$$

QED

Ruin Probability when the Claim Size is Exponentially Distributed.

Suppose the claim sizeis exponentially distributed that is $X \sim EXP\left(\alpha^{-}\right)$, then in order to obtain the probability of ruin, it is easier to consider the non-ruinintegral equation as follows

$$\frac{d\phi(u)}{du}=\frac{\lambda}{\pi}\phi(u)-\frac{\lambda}{\pi}\int_0^u\phi(u-x)f_X(x)dx \tag{73}$$

$$Z=u-x\Rightarrow dZ=-dx \tag{74}$$

$$\frac{d\phi(u)}{du}=\frac{\lambda}{\pi}\phi(u)-\frac{\lambda}{\alpha\pi}\int_0^u\phi(x)e^{-\frac{(u-x)}{\alpha}}dx \tag{75}$$

$$\frac{d\phi(u)}{du}=\frac{\lambda}{\pi}\phi(u)-\frac{\lambda}{\alpha\pi}e^{-\frac{u}{\alpha}}\int_0^u\phi(x)e^{\frac{x}{\alpha}}dx \tag{76}$$

$$\frac{d^2\phi(u)}{du^2}=\frac{\lambda}{\pi}\frac{d\phi(u)}{du}+\frac{1}{\alpha}\left(\frac{\lambda}{\pi}\phi(u)-\frac{d\phi(u)}{du}\right)-\frac{\lambda}{\pi\alpha}\phi(u) \tag{77}$$

$$\frac{d^2\phi(u)}{du^2}=\frac{\lambda}{\pi}\frac{d\phi(u)}{du}-\frac{1}{\alpha}\frac{d\phi(u)}{du}+\frac{1}{\alpha}\frac{\lambda}{\pi}\phi(u)-\frac{\lambda}{\pi\alpha}\phi(u) \tag{78}$$

$$\frac{d^2\phi(u)}{du^2}=\left(\frac{\lambda}{\pi}-\frac{1}{\alpha}\right)\frac{d\phi(u)}{du}=-\frac{\theta}{\alpha(1+\theta)}\frac{d\phi(u)}{du} \tag{79}$$

$$\frac{d^2\phi(u)}{du^2}=\left(\frac{\alpha\lambda-1}{\alpha\pi}\right)\frac{d\phi(u)}{du}=\left(\frac{\frac{\alpha\lambda}{\pi}-1}{\alpha}\right)\frac{d\phi(u)}{du} \tag{80}$$

where

$$\frac{\alpha\lambda}{\pi}=\frac{1}{1+\theta} \tag{81}$$

$$\frac{d^2\phi(u)}{du^2}=\left(\frac{\frac{1}{1+\theta}-1}{\alpha}\right)\frac{d\phi(u)}{du}=\left(\frac{\frac{1-1-\theta}{1+\theta}}{\alpha}\right)\frac{d\phi(u)}{du} \tag{82}$$

$$\frac{d^2\phi(u)}{du^2}=\left(\frac{\frac{-\theta}{(1+\theta)}}{\alpha}\right)\frac{d\phi(u)}{du}=\frac{-\theta}{\alpha(1+\theta)}\frac{d\phi(u)}{du} \tag{83}$$

$$\frac{d^2\phi(u)}{du^2}=\frac{-\theta u}{\alpha(1+\theta)}\frac{d\phi(u)}{du}\Rightarrow\frac{\phi''(u)}{\phi'(u)}=-\frac{\theta u}{\alpha(1+\theta)}=\frac{d}{du}\left(\log_e\phi'(u)\right) \tag{84}$$

$$\log_e\left[\frac{d\phi(u)}{du}\right]=\frac{-\theta u}{\alpha(1+\theta)}+\kappa_1 \tag{85}$$

$$\frac{d\phi(u)}{du}=\kappa_2\exp\left(\frac{-\theta u}{\alpha(1+\theta)}\right)\Rightarrow\phi(u)=\kappa_3\exp\left(\frac{-\theta u}{\alpha(1+\theta)}\right)+\kappa_4 \tag{86}$$

By the monotone convergence theorem, we have

$$\phi(\infty)=1\Rightarrow\kappa_4=1 \tag{86a}$$

$$\phi(0)=1-\frac{1}{(1+\theta)}\Rightarrow\kappa_3=-\frac{1}{(1+\theta)} \tag{87}$$

$$\phi(u)=1-\frac{1}{1+\theta}e^{\left(\frac{-\theta u}{\alpha(1+\theta)}\right)} \tag{88}$$

Therefore, the ruin probability is given as

$$\psi(u)=1-\phi(u)=\frac{1}{1+\theta}e^{\left(\frac{-\theta u}{\alpha(1+\theta)}\right)} \tag{89}$$

$$\psi(0)=\frac{1}{1+\theta} \tag{89a}$$

Theorem

Suppose that the ruin probability $\psi(u)$ is differentiable, then within the interval $0<u<\frac{\pi}{\delta}$ where $0<\delta<1$,

$$(\delta u-\pi)\psi^{(1)}(u)=-\lambda\int_u^\infty S_X(\xi-u)H(\xi)d\xi \tag{90a}$$

$$\psi^{(2)}(u)+\psi^{(1)}(u)=\left[\frac{\begin{Bmatrix}\lambda H(u)\\ +\int_u^\infty \lambda h(\xi-u)S_X(\xi-u)H(\xi)-\lambda S_X(\xi-u)H(\xi)d\xi\end{Bmatrix}}{(\delta u-\pi)}\right] \tag{90b}$$

Proof:

Let $f_X(x)$, $S_X(x)$ and $F_X(x)$ be the probability density and survival functions and distribution function respectively. $\psi^{(1)}(u)$ and $\psi^{(2)}(u)$ are the first and second order co-efficients with respect to argument u

Observe that the ruin probability satisfies theintegro-differential equation defined by

$$(\delta u-\pi)\psi'(u)=\lambda\psi(u)-\lambda\int_0^{\frac{\pi}{\delta}-u} f_X(x)\psi(u+x)dx \tag{91}$$

But

$$\int_0^\infty f_X(x)\psi(u)dx=\psi(u)\int_0^\infty f_X(x)dx \tag{92}$$

$$\int_0^\infty f_X(x)\psi(u)dx=\psi(u).1=\psi(u) \tag{93}$$

again

$$\int_0^\infty f_X(x)\psi(u)dx=\int_0^{\frac{\pi}{\delta}-u} f_X(x)\psi(u)dx+\int_{\frac{\pi}{\delta}-u}^\infty f_X(x)\psi(u)dx \tag{94}$$

Therefore

$$\lambda\psi(u)-\lambda\int_0^{\frac{\pi}{\delta}-u} f_X(x)\psi(u+x)dx= \\ \lambda\int_0^{\frac{\pi}{\delta}-u} f_X(x)\psi(u)dx-\lambda\int_0^{\frac{\pi}{\delta}-u} f_X(x)\psi(u+x)dx+\lambda\int_{\frac{\pi}{\delta}-u}^\infty f_X(x)\psi(u)dx \tag{95}$$

Let the function $H(.)$ defined in equations (90a) and (90b)

$$H(u)=\begin{cases}\frac{d}{du}\psi(u) & for \quad 0\le u\le\frac{\pi}{\delta}\\ 0 & for \quad u>\frac{\pi}{\delta}\end{cases} \tag{96}$$

Recall that the initial capital satisfies the inequality $0\le u\le\frac{\pi}{\delta}$, consequently we have

$$\lambda\psi(u)-\lambda\int_0^{\frac{\pi}{\delta}} f_X(x)\psi(u+x)dx=\lambda\int_0^\infty f_X(x)\int_{u+x}^u H(\xi)d\xi dx \tag{97}$$

$$\lambda\psi(u)-\lambda\int_0^{\frac{\pi}{\delta}} f_X(x)\psi(u+x)dx=\lambda\int_0^\infty f_X(x)\int_{u+x}^u \psi'(\xi)d\xi dx \tag{98}$$

$$\lambda\psi(u)-\lambda\int_0^{\frac{\pi}{\delta}} f_X(x)\psi(u+x)dx=\lambda\int_0^\infty f_X(x)\left[\psi(\xi)\right]_{u+x}^u dx \tag{99}$$

$$\lambda\psi(u)-\lambda\int_0^{\frac{\pi}{\delta}} f_X(x)\psi(u+x)dx=\lambda\int_0^\infty f_X(x)\left[\psi(u)-\psi(u+x)\right]dx \tag{100}$$

By changing the order of integration we have

$$\lambda\psi(u)-\lambda\int_0^{\frac{\pi}{\delta}} f_X(x)\psi(u+x)\,dx=-\lambda\int_u^\infty\int_{\xi=u}^\infty f_X(x)\,dx H(\xi)\,d\xi \tag{101}$$

$$\lambda\psi(u)-\lambda\int_0^{\frac{\pi}{\delta}} f_X(x)\psi(u+x)\,dx=-\lambda\int_u^\infty S_X(\xi-u)H(\xi)\,d\xi \tag{102}$$

$$(\delta u-\pi)\psi^{(1)}(u)=-\lambda\int_u^\infty S_X(\xi-u)H(\xi)\,d\xi \tag{103}$$

$$(\delta u-\pi)\psi^{(2)}(u)=-\lambda\left[-S_X(0)H(u)+\int_u^\infty \frac{\partial}{\partial u}S_X(\xi-u)H(\xi)\,d\xi\right] \tag{104}$$

$$(\delta u-\pi)\psi^{(2)}(u)=\left[\lambda H(u)-\lambda\int_u^\infty \frac{\partial}{\partial u}S_X(\xi-u)H(\xi)\,d\xi\right] \tag{105}$$

$$(\delta u-\pi)\psi^{(2)}(u)=\lambda H(u)-\lambda\int_u^\infty S'_X(\xi-u)H(\xi)\,d\xi \tag{106}$$

By reason of the differential co-efficient under the integral operator in equations (104), (105) and (106), we differentiate the distribution function. Therefore, differentiating the distribution function $F_X(x)$, we have the probability density function $f_X(x)$,

$$\frac{d}{dx}F_X(x)=f_X(x) \tag{107}$$

Recall that the survival and the distribution function of the claim size X add up to 1

$$S_X(x)+F_X(x)=1 \tag{108}$$

Therefore, using equation (108), we obtain

$$\frac{d}{dx}S_X(x)+f_X(x)=0 \tag{109}$$

$$\frac{d}{dx}S_X(x)=-f_X(x) \tag{110}$$

Substituting equation (110) in equation (106), we obtain

$$(\delta u-\pi)\psi^{(2)}(u)=\lambda H(u)-\lambda\int_u^\infty(-1)f_X(\xi-u)H(\xi)\,d\xi \tag{111}$$

$$(\delta u-\pi)\psi^{(2)}(u)=\lambda H(u)+\lambda\int_u^\infty f_X(\xi-u)H(\xi)\,d\xi \tag{112}$$

$$(\delta u-\pi)\psi^{(2)}(u)+(\delta u-\pi)\psi^{(1)}(u)=\lambda H(u)+\lambda\int_u^\infty f_X(\xi-u)H(\xi)\,d\xi-\lambda\int_u^\infty S_X(\xi-u)H(\xi)\,d\xi \tag{113}$$

$$\psi^{(2)}(u)+\psi^{(1)}(u)=\frac{1}{(\delta u-\pi)}\begin{bmatrix}\lambda H(u)\\ +\lambda\int_u^\infty f_X(\xi-u)H(\xi)\,d\xi-\lambda\int_u^\infty S_X(\xi-u)H(\xi)\,d\xi\end{bmatrix} \tag{114}$$

But the hazard function is

$$h(\xi-u)=\frac{f_X(\xi-u)}{S_X(\xi-u)} \tag{115}$$

$$\psi^{(2)}(u)+\psi^{(1)}(u)=\left[\frac{\begin{Bmatrix}\lambda H(u)\\ +\lambda\int_u^\infty h(\xi-u)S_X(\xi-u)H(\xi)\,d\xi-\lambda\int_u^\infty S_X(\xi-u)H(\xi)\,d\xi\end{Bmatrix}}{(\delta u-\pi)}\right] \tag{116}$$

$$\psi^{(2)}(u)+\psi^{(1)}(u)-\left[\frac{\begin{Bmatrix}\lambda H(u)\\ +\int_u^\infty \lambda h(\xi-u)S_X(\xi-u)H(\xi)-\lambda S_X(\xi-u)H(\xi)d\xi\end{Bmatrix}}{(\delta u-\pi)}\right]=0 \tag{117}$$

QED

Material and Methods

Numerical Estimation of the Adjustment Co-efficient

The adjustment co-efficient R in equations (45a), (47), (48) and (53) cannot be analytically determined unless it is numerically estimated to enable us estimate probability of ruin under the chosen models.

$$M_X(R)-1=\frac{\pi R}{\lambda} \tag{118}$$

$$\lambda+\pi R=\lambda M_X(R) \tag{119}$$

$$\lambda+\pi R=\lambda\int_0^\infty e^{Rx}f_X(x)dx \tag{120}$$

$$\lambda+\pi R=\lambda\int_0^\infty\left(1+Rx+\frac{R^2x^2}{2}+\frac{R^3x^3}{3}+\frac{R^4x^4}{4}+\cdots\right)f_X(x)dx \tag{121}$$

Ignoring the fifth and higher terms, we have

$$\lambda+\pi R\approx\lambda\int_0^\infty\left(1+Rx+\frac{R^2x^2}{2}+\frac{R^3x^3}{3}\right)f_X(x)dx \tag{122}$$

$$\lambda+\pi R\approx\lambda\left\{\int_0^\infty f_X(x)dx+R\int_0^\infty xf_X(x)dx+\frac{R^2}{2}\int_0^\infty x^2f_X(x)dx+\frac{R^3}{3}\int_0^\infty x^3f_X(x)dx\right\} \tag{123}$$

$$\lambda+\pi R=\lambda\left[1+R\times\mathbf{E}(X)+\frac{R^2}{2}\mathbf{E}(X^2)+\frac{R^3}{3}\mathbf{E}(X^3)\right] \tag{124}$$

$$\lambda+\pi R=\left[\lambda+\lambda R\mathbf{E}(X)+\frac{\lambda R^2}{2}\mathbf{E}(X^2)+\frac{\lambda R^3}{3}\mathbf{E}(X^3)\right] \tag{125}$$

$$\pi R=\lambda R\mathbf{E}(X)+\frac{\lambda R^2}{2}\mathbf{E}(X^2)+\frac{\lambda R^3}{3}\mathbf{E}(X^3) \tag{126}$$

$$\pi=\lambda\mathbf{E}(X)+\frac{\lambda R}{2}\mathbf{E}(X^2)+\frac{\lambda R^2}{3}\mathbf{E}(X^3) \tag{127}$$

But

$$\pi=(1+\theta)\lambda\mathbf{E}(X) \tag{128}$$

$$(1+\theta)\lambda\mathbf{E}(X)=\frac{\lambda R^2}{3}\mathbf{E}(X^3)+\frac{\lambda R}{2}\mathbf{E}(X^2)+\lambda\mathbf{E}(X) \tag{129}$$

$$2\lambda\mathbf{E}(X^3)R^2+3\lambda\mathbf{E}(X^2)R+6\lambda\mathbf{E}(X)-6(1+\theta)\lambda\mathbf{E}(X)=0 \tag{130}$$

$$2\lambda\mathbf{E}(X^3)R^2+3\lambda\mathbf{E}(X^2)R+6\lambda\mathbf{E}(X)-6\lambda\mathbf{E}(X)-6\lambda\theta\mathbf{E}(X)=0 \tag{131}$$

$$2\lambda\mathbf{E}\left(X^3\right)R^2+3\lambda\mathbf{E}\left(X^2\right)R-6\lambda\theta\mathbf{E}\left(X\right)=0 \tag{132}$$

$$R=\frac{-3\lambda\mathbf{E}\left(X^2\right)\pm\sqrt{\left(3\lambda\mathbf{E}\left(X^2\right)\right)^2+48\lambda^2\theta\times\mathbf{E}\left(X\right)\times\mathbf{E}\left(X^3\right)}}{4\lambda\mathbf{E}\left(X^3\right)} \tag{133}$$

$$R=\frac{-3\lambda\mathbf{E}\left(X^2\right)+\sqrt{\left(3\lambda\mathbf{E}\left(X^2\right)\right)^2+48\lambda^2\theta\times\mathbf{E}\left(X\right)\times\mathbf{E}\left(X^3\right)}}{4\lambda\mathbf{E}\left(X^3\right)} \tag{134}$$

$$R=\frac{-3\lambda\mathbf{E}\left(X^2\right)+\sqrt{9\lambda^2\left(\mathbf{E}\left(X^2\right)\right)^2+48\lambda^2\theta\times\mathbf{E}\left(X\right)\times\mathbf{E}\left(X^3\right)}}{4\lambda\mathbf{E}\left(X^3\right)} \tag{135}$$

$$R=\frac{\sqrt{9\lambda^2\mu_2^2+48\lambda^2\theta\mu\mu_3}-3\mu_2}{4\mu_3} \tag{136}$$

The Tijim's approximation to ruin probability

$$\psi_T\left(u\right)=\left(\frac{1}{1+\theta}-A\right)e^{-\frac{u}{\alpha}}+Ae^{-Ru} \tag{137}$$

$$\alpha=\frac{\frac{E\left(X^2\right)}{2\mu\theta}-\frac{A}{R}}{\frac{1}{\left(1+\theta\right)}-A} \tag{138}$$

$\theta=$ Loading and $\mu=\mathbf{E}\left(X\right)$

Since

$$\frac{\lambda\mu}{\pi}=\frac{1}{\left(1+\theta\right)} \tag{139}$$

$$A=\frac{\mu\theta}{M'_X\left(R\right)-\mu\left(1+\theta\right)} \tag{140}$$

A is from the Cramer's asymptotic approximation

The De-Vylder's Approximation

The rationale behind this model is to replace the surplus process with one with $\overline{\theta}=\theta$, $\overline{\lambda}=\lambda$ and exponential claims with parameters $\overline{\beta}=\beta$

$$\psi_D\left(u\right)=\frac{1}{1+\overline{\theta}}e^{-\frac{\overline{\theta}\,\overline{\beta}u}{1+\overline{\theta}}} \tag{141}$$

where

$$\overline{\theta}=\frac{2\mu\mu_3\theta}{3\left(\mu_2\right)^2} \tag{142}$$

$$\overline{\beta}=\frac{3\left(\mu_2\right)}{\left(\mu_3\right)} \tag{143}$$

$$\overline{\lambda} = \frac{9\lambda\left(\mu_2\right)^3}{2\left(\mu_3\right)^2} \qquad (144)$$

Data Analysis and Presentation

The Adjustment Coefficient

Table 1

The adjustment coefficient R when the values of safety loading are $0.1,\ 0.2\ and\ 0.3$

R when θ = 0.1	R when θ = 0.2	R when θ = 0.3
0.00000002387969	0.00000004215254	0.00000005754026

The result from table 1 showed that the adjustment coefficient increases as the safety loading increases.The table below shows the Tijim's ruin approximation to ruin probability $\Psi_T(u)$ for initial capital when the values of premium loading θ are $0.1,\ 0.2\ and\ 0.3$ respectively

Table 2

Tijim's ruin approximation

INITIAL CAPITAL (u)	$\psi_D(u)$; θ = 0.1	$\psi_D(u)$; θ = 0.2	$\psi_D(u)$; θ = 0.3
11000000	0.68826922	0.5044192	0.38691274
11100000	0.68652444	0.5021012	0.38446642
11200000	0.68478397	0.49979345	0.38203482
11300000	0.6830478	0.49749588	0.37961784
11400000	0.68131592	0.49520846	0.3772154
11500000	0.67958833	0.49293115	0.37482742
11600000	0.677865	0.4906639	0.3724538
11700000	0.67614594	0.48840667	0.37009447
11800000	0.67443112	0.48615941	0.36774934
11900000	0.67272055	0.48392209	0.36541833
12000000	0.6710142	0.48169465	0.36310135
12100000	0.66931207	0.47947706	0.36079833
12200000	0.66761415	0.47726927	0.35850917
12300000	0.66592043	0.47507124	0.3562338
12400000	0.6642309	0.47288293	0.35397214
12500000	0.66254555	0.47070429	0.3517241
12600000	0.66086436	0.46853529	0.34948962
12700000	0.65918733	0.46637589	0.3472686
12800000	0.65751445	0.46422603	0.34506096
12900000	0.6558457	0.46208568	0.34286664
13000000	0.65418109	0.45995481	0.34068554
13100000	0.65252059	0.45783336	0.3385176
13200000	0.65086419	0.45572129	0.33636274
13300000	0.6492119	0.45361858	0.33422087
13400000	0.64756369	0.45152517	0.33209192
13500000	0.64591956	0.44944102	0.32997582
13600000	0.64427949	0.4473661	0.32787249
13700000	0.64264349	0.44530036	0.32578186
13800000	0.64101153	0.44324378	0.32370385
13900000	0.63938361	0.44119629	0.32163838
14000000	0.63775971	0.43915788	0.31958538
14100000	0.63613984	0.43712849	0.31754479
14200000	0.63452397	0.43510809	0.31551652
14300000	0.6329121	0.43309663	0.3135005
14400000	0.63130422	0.43109409	0.31149667
14500000	0.62970031	0.42910042	0.30950494

14600000	0.62810038	0.42711559	0.30752525
14700000	0.62650441	0.42513955	0.30555753
14800000	0.62491238	0.42317227	0.30360171
14900000	0.6233243	0.4212137	0.30165772
15000000	0.62174015	0.41926382	0.29972548
15100000	0.62015992	0.41732258	0.29780493
15200000	0.6185836	0.41538995	0.29589599
15300000	0.61701118	0.41346589	0.29399861
15400000	0.61544265	0.41155036	0.29211271
15500000	0.61387801	0.40964332	0.29023823
15600000	0.61231724	0.40774474	0.28837509
15700000	0.61076033	0.40585459	0.28652324
15800000	0.60920728	0.40397282	0.2846826
15900000	0.60765808	0.40209939	0.2828531
16000000	0.60611271	0.40023428	0.28103469
16100000	0.60457116	0.39837745	0.2792273
16200000	0.60303344	0.39652886	0.27743085
16300000	0.60149952	0.39468847	0.2756453
16400000	0.5999694	0.39285625	0.27387057
16500000	0.59844307	0.39103216	0.2721066
16600000	0.59692051	0.38921618	0.27035332
16700000	0.59540173	0.38740825	0.26861068
16800000	0.59388672	0.38560836	0.26687861
16900000	0.59237545	0.38381646	0.26515704
17000000	0.59086793	0.38203251	0.26344593
17100000	0.58936414	0.3802565	0.26174519
17200000	0.58786407	0.37848837	0.26005478
17300000	0.58636773	0.3767281	0.25837463
17400000	0.58487509	0.37497565	0.25670468
17500000	0.58338614	0.37323098	0.25504488
17600000	0.58190089	0.37149407	0.25339515
17700000	0.58041931	0.36976489	0.25175545
17800000	0.57894141	0.36804338	0.25012571
17900000	0.57746717	0.36632954	0.24850587
18000000	0.57599658	0.36462331	0.24689588
18100000	0.57452963	0.36292467	0.24529568
18200000	0.57306632	0.36123358	0.24370521
18300000	0.57160664	0.35955002	0.24212441
18400000	0.57015058	0.35787394	0.24055322
18500000	0.56869812	0.35620532	0.23899159
18600000	0.56724926	0.35454413	0.23743946
18700000	0.56580399	0.35289032	0.23589678
18800000	0.56436231	0.35124388	0.23436349
18900000	0.5629242	0.34960476	0.23283954
19000000	0.56148965	0.34797294	0.23132486
19100000	0.56005866	0.34634838	0.2298194
19200000	0.55863122	0.34473106	0.22832312
19300000	0.55720732	0.34312094	0.22683595
19400000	0.55578694	0.34151798	0.22535785
19500000	0.55437009	0.33992217	0.22388875
19600000	0.55295675	0.33833346	0.2224286
19700000	0.55154692	0.33675183	0.22097736
19800000	0.55014058	0.33517725	0.21953497
19900000	0.54873772	0.33360968	0.21810137
20000000	0.54733835	0.3320491	0.21667652
20100000	0.54594245	0.33049547	0.21526037
20200000	0.54455001	0.32894877	0.21385285
20300000	0.54316102	0.32740896	0.21245393
20400000	0.54177548	0.32587601	0.21106354
20500000	0.54039337	0.3243499	0.20968164

20600000	0.53901469	0.3228306	0.20830819
20700000	0.53763943	0.32131807	0.20694312
20800000	0.53626758	0.31981228	0.20558639
20900000	0.53489914	0.31831321	0.20423795
21000000	0.53353409	0.31682083	0.20289776
21100000	0.53217243	0.31533511	0.20156575
21200000	0.53081414	0.31385602	0.20024189
21300000	0.52945923	0.31238353	0.19892612
21400000	0.52810767	0.31091761	0.19761841
21500000	0.52675947	0.30945823	0.19631869
21600000	0.52541462	0.30800536	0.19502692
21700000	0.5240731	0.30655899	0.19374306
21800000	0.52273491	0.30511907	0.19246706
21900000	0.52140004	0.30368558	0.19119887
22000000	0.52006849	0.30225849	0.18993844
22100000	0.51874024	0.30083778	0.18868574
22200000	0.51741528	0.29942342	0.18744071
22300000	0.51609362	0.29801537	0.18620331
22400000	0.51477523	0.29661361	0.18497349
22500000	0.51346012	0.29521812	0.18375121
22600000	0.51214827	0.29382887	0.18253642
22700000	0.51083968	0.29244582	0.18132908
22800000	0.50953434	0.29106896	0.18012915
22900000	0.50823224	0.28969825	0.17893658
23000000	0.50693337	0.28833368	0.17775133
23100000	0.50563772	0.2869752	0.17657335
23200000	0.5043453	0.2856228	0.1754026
23300000	0.50305608	0.28427645	0.17423903
23400000	0.50177006	0.28293612	0.17308262
23500000	0.50048724	0.28160179	0.17193331
23600000	0.4992076	0.28027343	0.17079105
23700000	0.49793114	0.27895101	0.16965582
23800000	0.49665785	0.27763451	0.16852756
23900000	0.49538772	0.27632391	0.16740624
24000000	0.49412075	0.27501917	0.16629182
24100000	0.49285692	0.27372027	0.16518425
24200000	0.49159623	0.27242719	0.16408349
24300000	0.49033868	0.2711399	0.1629895
24400000	0.48908424	0.26985838	0.16190225
24500000	0.48783293	0.2685826	0.16082169
24600000	0.48658472	0.26731253	0.15974778
24700000	0.48533961	0.26604815	0.15868048
24800000	0.4840976	0.26478944	0.15761976
24900000	0.48285868	0.26353637	0.15656557
25000000	0.48162283	0.26228891	0.15551788
25100000	0.48039005	0.26104705	0.15447664
25200000	0.47916034	0.25981076	0.15344183
25300000	0.47793368	0.25858	0.15241339
25400000	0.47671007	0.25735477	0.1513913
25500000	0.4754895	0.25613503	0.15037551
25600000	0.47427196	0.25492077	0.149366
25700000	0.47305745	0.25371195	0.14836271
25800000	0.47184596	0.25250855	0.14736561
25900000	0.47063748	0.25131056	0.14637467
26000000	0.46943201	0.25011794	0.14538985
26100000	0.46822953	0.24893067	0.14441112
26200000	0.46703004	0.24774874	0.14343843
26300000	0.46583353	0.2465721	0.14247175
26400000	0.46464	0.24540076	0.14151105
26500000	0.46344944	0.24423467	0.14055628

26600000	0.46226183	0.24307382	0.13960743
26700000	0.46107718	0.24191818	0.13866444
26800000	0.45989547	0.24076774	0.13772728
26900000	0.4587167	0.23962246	0.13679593
27000000	0.45754087	0.23848233	0.13587034
27100000	0.45636795	0.23734733	0.13495048
27200000	0.45519796	0.23621742	0.13403632
27300000	0.45403087	0.2350926	0.13312782
27400000	0.45286669	0.23397283	0.13222495
27500000	0.4517054	0.23285809	0.13132768
27600000	0.450547	0.23174837	0.13043596
27700000	0.44939149	0.23064364	0.12954978
27800000	0.44823884	0.22954388	0.12866909
27900000	0.44708907	0.22844907	0.12779387
28000000	0.44594215	0.22735918	0.12692407
28100000	0.44479809	0.2262742	0.12605968
28200000	0.44365688	0.2251941	0.12520065
28300000	0.4425185	0.22411887	0.12434695
28400000	0.44138296	0.22304847	0.12349856
28500000	0.44025024	0.22198289	0.12265544
28600000	0.43912034	0.22092211	0.12181756
28700000	0.43799325	0.21986611	0.12098488
28800000	0.43686897	0.21881486	0.12015738
28900000	0.43574749	0.21776835	0.11933503
29000000	0.4346288	0.21672656	0.11851779
29100000	0.43351289	0.21568945	0.11770564
29200000	0.43239976	0.21465703	0.11689854
29300000	0.4312894	0.21362925	0.11609647
29400000	0.4301818	0.21260611	0.11529939
29500000	0.42907696	0.21158758	0.11450728
29600000	0.42797488	0.21057365	0.1137201
29700000	0.42687553	0.20956428	0.11293782
29800000	0.42577892	0.20855947	0.11216042
29900000	0.42468505	0.20755919	0.11138787
30000000	0.42359389	0.20656343	0.11062014

Ruin Probability Using De-Vylder's Approximation Based on the Values of the Adjustment Coefficient *R*

The De-Vylder's approximation to ruin probability $\Psi_D(u)$ for initial capital u when the values of premium loading θ are 0.1, 0.2 and 0.3 respectively.

Table 3

The De-Vylder's approximation to ruin probability

INITIAL CAPITAL (u)	$\psi_D(u)$; $\theta = 0.1$	$\psi_D(u)$; $\theta = 0.2$	$\psi_D(u)$ $\theta = 0.3$
11000000	0.66905584	0.48013436	0.36286841
11100000	0.66734262	0.47792727	0.36059877
11200000	0.6656338	0.47573032	0.35834332
11300000	0.66392935	0.47354346	0.35610199
11400000	0.66222927	0.47136666	0.35387467
11500000	0.66053354	0.46919987	0.35166128
11600000	0.65884215	0.46704304	0.34946173
11700000	0.65715509	0.46489612	0.34727595
11800000	0.65547235	0.46275907	0.34510383
11900000	0.65379393	0.46063185	0.3429453
12000000	0.65211979	0.4585144	0.34080027
12100000	0.65044995	0.45640668	0.33866866
12200000	0.64878438	0.45430866	0.33655038
12300000	0.64712308	0.45222028	0.33444535
12400000	0.64546603	0.4501415	0.33235349
12500000	0.64381322	0.44807227	0.33027471

12600000	0.64216465	0.44601256	0.32820893
12700000	0.6405203	0.44396232	0.32615608
12800000	0.63888016	0.4419215	0.32411606
12900000	0.63724422	0.43989006	0.3220888
13000000	0.63561246	0.43786796	0.32007423
13100000	0.63398489	0.43585515	0.31807225
13200000	0.63236148	0.4338516	0.3160828
13300000	0.63074223	0.43185726	0.31410578
13400000	0.62912713	0.42987208	0.31214114
13500000	0.62751616	0.42789603	0.31018878
13600000	0.62590932	0.42592907	0.30824864
13700000	0.62430659	0.42397114	0.30632063
13800000	0.62270796	0.42202222	0.30440468
13900000	0.62111343	0.42008225	0.30250071
14000000	0.61952299	0.41815121	0.30060865
14100000	0.61793661	0.41622904	0.29872843
14200000	0.6163543	0.4143157	0.29685996
14300000	0.61477604	0.41241116	0.29500318
14400000	0.61320182	0.41051538	0.29315802
14500000	0.61163163	0.40862831	0.2913244
14600000	0.61006546	0.40674991	0.28950224
14700000	0.6085033	0.40488015	0.28769149
14800000	0.60694514	0.40301898	0.28589206
14900000	0.60539097	0.40116637	0.28410388
15000000	0.60384079	0.39932228	0.28232689
15100000	0.60229457	0.39748666	0.28056101
15200000	0.60075231	0.39565948	0.27880618
15300000	0.599214	0.3938407	0.27706232
15400000	0.59767963	0.39203028	0.27532938
15500000	0.59614918	0.39022819	0.27360727
15600000	0.59462266	0.38843437	0.27189593
15700000	0.59310005	0.38664881	0.27019529
15800000	0.59158133	0.38487145	0.2685053
15900000	0.5900665	0.38310226	0.26682587
16000000	0.58855556	0.3813412	0.26515695
16100000	0.58704848	0.37958824	0.26349846
16200000	0.58554526	0.37784334	0.26185035
16300000	0.58404589	0.37610645	0.26021255
16400000	0.58255036	0.37437756	0.25858499
16500000	0.58105866	0.37265661	0.25696762
16600000	0.57957077	0.37094357	0.25536035
16700000	0.5780867	0.3692384	0.25376315
16800000	0.57660643	0.36754107	0.25217593
16900000	0.57512995	0.36585155	0.25059864
17000000	0.57365725	0.36416979	0.24903121
17100000	0.57218832	0.36249577	0.24747359
17200000	0.57072315	0.36082943	0.24592571
17300000	0.56926173	0.35917076	0.24438751
17400000	0.56780406	0.35751971	0.24285894
17500000	0.56635012	0.35587626	0.24133992
17600000	0.5648999	0.35424035	0.23983041
17700000	0.56345339	0.35261197	0.23833033
17800000	0.56201059	0.35099107	0.23683964
17900000	0.56057149	0.34937762	0.23535828
18000000	0.55913606	0.34777159	0.23388617
18100000	0.55770432	0.34617295	0.23242328
18200000	0.55627624	0.34458165	0.23096954
18300000	0.55485182	0.34299766	0.22952489
18400000	0.55343104	0.34142096	0.22808927
18500000	0.5520139	0.33985151	0.22666264

18600000	0.55060039	0.33828927	0.22524492
18700000	0.5491905	0.33673421	0.22383608
18800000	0.54778423	0.3351863	0.22243605
18900000	0.54638155	0.3336455	0.22104477
19000000	0.54498246	0.33211179	0.21966219
19100000	0.54358696	0.33058513	0.21828827
19200000	0.54219503	0.32906549	0.21692294
19300000	0.54080666	0.32755283	0.21556614
19400000	0.53942185	0.32604712	0.21421784
19500000	0.53804058	0.32454834	0.21287796
19600000	0.53666286	0.32305644	0.21154647
19700000	0.53528866	0.32157141	0.2102233
19800000	0.53391797	0.3200932	0.20890841
19900000	0.5325508	0.31862178	0.20760175
20000000	0.53118713	0.31715713	0.20630326
20100000	0.52982695	0.31569922	0.20501289
20200000	0.52847026	0.314248	0.20373059
20300000	0.52711704	0.31280345	0.20245631
20400000	0.52576728	0.31136555	0.20119
20500000	0.52442098	0.30993425	0.19993161
20600000	0.52307812	0.30850954	0.1986811
20700000	0.52173871	0.30709137	0.1974384
20800000	0.52040273	0.30567972	0.19620348
20900000	0.51907016	0.30427457	0.19497628
21000000	0.51774101	0.30287587	0.19375676
21100000	0.51641526	0.3014836	0.19254486
21200000	0.51509291	0.30009773	0.19134055
21300000	0.51377394	0.29871823	0.19014376
21400000	0.51245835	0.29734507	0.18895447
21500000	0.51114613	0.29597823	0.18777261
21600000	0.50983727	0.29461767	0.18659814
21700000	0.50853176	0.29326336	0.18543102
21800000	0.50722959	0.29191528	0.1842712
21900000	0.50593076	0.29057339	0.18311864
22000000	0.50463525	0.28923767	0.18197328
22100000	0.50334306	0.2879081	0.18083509
22200000	0.50205418	0.28658463	0.17970401
22300000	0.5007686	0.28526725	0.17858002
22400000	0.49948632	0.28395592	0.17746305
22500000	0.49820731	0.28265063	0.17635306
22600000	0.49693158	0.28135133	0.17525002
22700000	0.49565912	0.28005801	0.17415388
22800000	0.49438991	0.27877063	0.1730646
22900000	0.49312396	0.27748916	0.17198213
23000000	0.49186125	0.27621359	0.17090643
23100000	0.49060177	0.27494389	0.16983745
23200000	0.48934551	0.27368002	0.16877517
23300000	0.48809248	0.27242195	0.16771953
23400000	0.48684265	0.27116968	0.16667049
23500000	0.48559602	0.26992316	0.16562801
23600000	0.48435258	0.26868236	0.16459205
23700000	0.48311233	0.26744728	0.16356257
23800000	0.48187525	0.26621787	0.16253953
23900000	0.48064134	0.26499411	0.16152289
24000000	0.47941059	0.26377597	0.16051261
24100000	0.478183	0.26256344	0.15950865
24200000	0.47695854	0.26135648	0.15851097
24300000	0.47573722	0.26015507	0.15751953
24400000	0.47451903	0.25895918	0.15653428
24500000	0.47330396	0.25776879	0.15555521

24600000	0.472092	0.25658387	0.15458225
24700000	0.47088314	0.2554044	0.15361538
24800000	0.46967738	0.25423034	0.15265456
24900000	0.4684747	0.25306169	0.15169975
25000000	0.46727511	0.25189841	0.15075091
25100000	0.46607859	0.25074047	0.149808
25200000	0.46488513	0.24958786	0.14887099
25300000	0.46369472	0.24844055	0.14793985
25400000	0.46250737	0.24729851	0.14701452
25500000	0.46132305	0.24616172	0.14609499
25600000	0.46014177	0.24503016	0.1451812
25700000	0.45896351	0.24390379	0.14427314
25800000	0.45778827	0.24278261	0.14337075
25900000	0.45661604	0.24166658	0.142474
26000000	0.45544681	0.24055568	0.14158287
26100000	0.45428058	0.23944988	0.14069731
26200000	0.45311733	0.23834917	0.13981728
26300000	0.45195706	0.23725352	0.13894276
26400000	0.45079976	0.23616291	0.13807372
26500000	0.44964542	0.23507731	0.1372101
26600000	0.44849404	0.2339967	0.13635189
26700000	0.44734561	0.23292105	0.13549905
26800000	0.44620012	0.23185035	0.13465154
26900000	0.44505756	0.23078458	0.13380933
27000000	0.44391793	0.2297237	0.13297239
27100000	0.44278122	0.2286677	0.13214068
27200000	0.44164741	0.22761655	0.13131418
27300000	0.44051651	0.22657024	0.13049285
27400000	0.43938851	0.22552873	0.12967665
27500000	0.43826339	0.22449201	0.12886556
27600000	0.43714116	0.22346006	0.12805954
27700000	0.4360218	0.22243285	0.12725856
27800000	0.4349053	0.22141037	0.12646259
27900000	0.43379167	0.22039258	0.12567161
28000000	0.43268088	0.21937947	0.12488556
28100000	0.43157294	0.21837102	0.12410444
28200000	0.43046784	0.21736721	0.1233282
28300000	0.42936557	0.21636801	0.12255682
28400000	0.42826612	0.2153734	0.12179026
28500000	0.42716948	0.21438336	0.12102849
28600000	0.42607566	0.21339788	0.12027149
28700000	0.42498463	0.21241693	0.11951923
28800000	0.4238964	0.21144048	0.11877167
28900000	0.42281095	0.21046853	0.11802878
29000000	0.42172828	0.20950104	0.11729055
29100000	0.42064839	0.208538	0.11655693
29200000	0.41957126	0.20757938	0.1158279
29300000	0.41849689	0.20662517	0.11510342
29400000	0.41742527	0.20567535	0.11438348
29500000	0.41635639	0.2047299	0.11366805
29600000	0.41529026	0.20378879	0.11295708
29700000	0.41422685	0.20285201	0.11225057
29800000	0.41316616	0.20191953	0.11154847
29900000	0.41210819	0.20099134	0.11085077
30000000	0.41105293	0.20006742	0.11015743

Results and Discussion

We recall from equation (43) that $\mathbf{E}\left(U\left(\xi\right)\right)=u+\pi\xi-\mu\lambda\xi$
But by the Waald's identity,

$$\lim_{\xi\to\infty}\left\{\frac{\mathbf{E}\left(U\left(\xi\right)\right)}{\xi}\right\}=\lim_{\xi\to\infty}\left(\frac{u+\pi\xi-\mu\lambda\xi}{\xi}\right)=\lim_{\xi\to\infty}\left(\frac{u}{\xi}+\frac{\pi\xi-\mu\lambda\xi}{\xi}\right) \quad (144a)$$

$$\lim_{\xi\to\infty}\left\{\frac{\mathbf{E}\left(U\left(\xi\right)\right)}{\xi}\right\}=\pi-\mu\lambda \quad (144b)$$

By the reason of the strong law of large numbers, it is clear that in the long run $U\left(\xi\right)$ will converge to $-\infty$ (a.s) when $\pi > \mu\lambda$ whereas if $\pi < \mu\lambda$ then $U\left(\xi\right)$ will converge to ∞ (a.s) for all ξ. However, if $\pi = \mu\lambda$, then $\liminf_{\xi\to\infty}\left\{U\left(\xi\right)\right\}=\lim_{\xi\to\infty} Sup\left\{U\left(\xi\right)\right\}=-\infty$.

The result from table 2 and 3suggests that the probability of ruin decreases as the initial capital increasessignifying that the higher the initial capital u the higher the survival probability of the model. It also shows that that the probability of ruin decreases as the safety loading θ increases indicatingthat the bigger the adjustment coefficient R the higher survival probability the model has.Table 1 shows the values of the adjustment coefficient R for the corresponding values of the safety loadings 0.1, 0.2 *and* 0.3 respectively. The adjustment coefficient therefore increases as the safety loading increases. The result from tables 2-3 showed that as the level of ruin probabilities decreases, the size of the initial capital increases establishing an inverse linear relationship and consequently. Thus a straight line could be applied to forecast future initial capital barring core changes in the risk parameters such as the safety loading and claim sizes. In the event of high level of ruin probabilities, then there will be an anticipated bigger risk appetite within the insurer's underwriting profile implying a small provision for contingency regarding insolvencies such that where the anticipated ruin probabilities declines, there will be a small risk perception adopted and therefore a bigger provision for the anticipated ruin. The stress analysis conducted in the Tables 2-3 reveals that the probability of ruin decreases as the safety loading θ increases implying that the higher the adjustment coefficient R, the higher the survival probability the models have. Computational evidence from tables 2-3 shows that the Tijim's approximation to ruin probability is higher than the De-Vylder's approximation at the same level of initial capital and safety loading. The implication is that the De-Vylder's approximation to ruin probabilities is an improvement over Tijim's ruin model and hence De-Vylder's approximation is recommended for the insurance firm. From the foregoing, the results obtained could be employed to advise the insurance firms through the regulatory authorities to enshrine policy framework which can forestall pervasive consequences of ruin and consequently, the regulatory authorities should therefore enforce policy recommendations on improved minimum capital to escape ruinous conditions.

Conclusion

From our discussions, it is necessary for an insurance firm to conduct underwriting business above a defined level of income assumed set above zero or a specified threshold. The time that ruin occurs is then $\xi = \textit{infimum}\left\{t>0\middle|U_t\le 0\right\}$ and consequently $\psi\left(u\right)=\mathbf{P}\left(\xi<\infty\middle|U_0=u\right)$. Therefore, the continuous time minimum initial capital requires that the surplus be closely examined to ascertain that ruin does not occur. In our computations, the performance of the approximation was checked by comparing the Tijim's and De-Vylder's ruin probabilities using real claims data from a Nigerian insurance company.From the computed results, a high adjustment coefficient reduces the ruin probability of an underwriter whereas a high initial capital increases the solvency of an insurance company.Verified results from our estimations, De-Vylder's ruin probabilities seems more reliable as the fundamental principle is to replace surplus process with the one characterized by exponentially distributed losses and such that the first three moments coincide. Consequently, the ruin probability using De-Vylder's model is less than that of the Tijim'sapproximation at the same level of initial capital and safety loading. A sufficient minimum insurance capital should be set by the management of an insurance company in order to ensure the solvency of the company. Againan adequate safety loading should also be advisedsuch that the company will not enter into ruin in a foreseeable long run.

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Conflict of Interest

The author(s) declared no potential conflictof interest with respect to the research, authorship and/or publication of this article.

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