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Numerical Solution of Time-fractional Reaction Diffusion Equation via Elzaki Transform with Residual Power Series Method

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Abstract

The Elzaki transform with residual power series method is an efficient and reliable approach for solution of linear and non-linear fractional order differential equations. The major purpose of current work is to find the solution of time-fractional reaction diffusion equation by Elzaki transform with residual power series method. Elzaki transform is applied on this equation and then inverse Elzaki is taken on same equation for finding the expression of series solution. Then, assumed approximate solution is substituted on considered equation and unidentified coefficient functions are obtained by using residual function is equal to zero as well as combining its initial circumstances. At last, coefficient functions are substituted in power series form for finding finite approximate analytical solutions. The comparison between exact solution and approximate analytic solutions with different number of terms of this equation are determined and observed for reliability. This method reduces the size of computational works of solution of fractional order reaction diffusion equation. This article is anonymously gives the idea of education in Mathematics in higher studies. The nineteenth-century development of fractional derivatives and integrals, however, began with this paper, which introduced them independently. Among the scientific and engineering domains where fractional calculus is widely utilised are chemistry, physics, economics, biology, and finance. Fifteen years on, fractional calculus has gained popularity because of its proven applicability in many scientific and technical sectors.

Keywords: Elzaki transforms, reaction diffusion equation, Caputo-fractional derivatives, inverse Elzaki transform

Introduction

By providing non-integer order integral and derivative, the field of mathematics known as Fractional calculus expands on classical calculus. It was first studied in the 17th century by Leibniz and L'Hopital, who looked into the idea of defining a derivative of non-integer order. However, the work of Liouville, Riemann, and Grunwald, who independently introduced fractional derivatives and integrals, marked the beginning of the current development of it in the nineteenth century (Kilbas et. al., 2006). Chemistry, physics, economics, biology, and finance are just a few of the scientific and engineering fields where fractional calculus is used extensively. It has been used, for instance, to represent intricate systems like electrical circuits, signal processing, and viscoelastic materials.

Due to its well-established applications in numerous scientific and technical fields, fractional calculus has become more and more popular over the past three decades. Fractional-order models can accurately represent complex occurrences when changed by integer-order models, as several pioneers have shown (Miller, 1993). The fractional system of FDEs has attracted much attention in recent decades because it can explain complex phenomena that cannot be explained by conventional integer-order models. Numerous disciplines, including Geo-Science, Biology, Economics, Physics and Engineering, frequently employ the fractional system of FDEs. For simulating and understanding complex systems with long-range dependence, memory, and fractal behaviour, it is a useful tool (Hu. et.al., 2019). We shall discuss the definition, characteristics, and uses of the fractional system of FDEs in this article. We also discuss the numerical techniques used to resolve these equations and a number of unresolved issues in this area (Akinyemi, 2021).

The behaviour of a large range of chemical systems where the diffusion of materials completes with production of materials by means of some forms of chemical reactions is described by the reaction diffusion equations. The simplest reaction diffusion models are of the form,

$$D_t^{\alpha} u(x,t) = M(x) D_x^2 u(x,t) + r(u) \text{ where } 0 < \alpha \leq 1 \quad (1)$$

Where $r(u)$ is a non-linear analytic function of u and it is chosen as reaction kinetics.

$M(x)$ = continuous function for $x \in R, t > 0$.

When time first derivative is replaced by time-fractional derivative the equation (1) becomes

$$D_t^{\alpha} u(x,t) = M(x) D_x^2 u(x,t) + r(u) \text{ where } 0 < \alpha \leq 1 \quad (2)$$

Here α is a parameter defining time Caputo fractional derivative order.

Consider the time-fractional reaction diffusion equation (TFRDE)

$D_t^{\alpha}u(x,t) = D_x^2u(x,t) + u(x,t)\{1-u(x,t)\}$ where $0 < \alpha \leq 1, x \in R$ (3)

Subject to initial condition $u(x,0) = \mu$ (4)

Also the exact solution is $u(x,t) = \frac{\mu e^t}{1-\mu+\mu e^t}$ (5)

The equation (3) is a TFRDE and it is solved by using ERPSM. There are so many reliable, popular as well as efficient numerical and analytic methods having more reliability for finding the solutions of fractional order differential problems. Fractional integrals as well as derivatives are the branch of applied mathematical analysis and have been urbanized theoretically in recent years (Poudley et.al., 1998) the purpose of this field has been used in many areas such as sciences, engineering, aerodynamic, thermodynamic, mechatronics, image processing, physics, and fluid flow phenomena (Alquram, 2019). The fundamental improvement of FDEs is to provide a tool for the classification of different behaviours in different fields. Most FDEs do not have exact analytic solutions, so approximation methods must be used, out of them residual power series method (Pant, 2024), homotopy perturbation and analysis methods (Yousef et. al., 2019), differential transform method (Odibat et. al., 2006), iterative method (Raj, 2006), Adomian decomposition method (Jaradat et.al., 2018), different forms of non-integer power series version (Yang, 2016), biological engineering image dispensation (Metzler, 2000), physical model (Edeki, 2014), risk analysis (El-Ajou et.al., 2015), Taylor's method (Eriquat, 2020) and so many other methods have been used for solving fractional order linear or non-linear (Komalyaskha, 2016) differential equations. The residual power series approach (Arora et.al., 2023) is also applied to solve various non-integer order differential equations in one or two dimensions (Pant et.al., 2024).

The manuscript has been set up as follows: Introduction is presented in the first section. The second section provides a detailed explanation of the methods used to solve the TFRDE using ERPSM. The numerical experiment of this equation is provided in the third section. The numerical simulations, graphs and discussions are presented in section four. The research topic's conclusion is finally discussed in the fifth section.

Methodology

The methodology of solution of TFRDE by using ERPSM is performed in following steps:

Step 1 Applying Elzaki transform on equation (3) as,

$E[D_t^{\alpha}u(x,t)] = E[D_x^2u(x,t) + u(x,t)\{1-u(x,t)\}]$ (6)

Applying the differentiation property of Elzaki transform,

$E[D_t^\alpha u(x,t)] = \frac{1}{v^\alpha}\{E\big(u(x,t)\big) - g(x,t)\}$ on equation (6), we get

$$\frac{1}{v^\alpha}\{E\big(u(x,t)\big) - g(x,t)\} = E[D_x^2 u(x,t) + u(x,t)\{1-u(x,t)\}]$$

$$i.e.\, E\big(u(x,t)\big) = g(x,t) + v^\alpha E[D_x^2 u(x,t) + u(x,t)\{1-u(x,t)\}] \quad (7)$$

Step 2 Taking inverse Elzaki transform in equation (7), we get

$$u(x,t) = G(x,t) + E^{-1}[v^\alpha E[D_x^2 u(x,t) + u(x,t)\{1-u(x,t)\}]] \quad (8)$$

where $G(x,t) = E^{-1}\{g(x,t)\}$.

Step 3 By this method algorithm of $u(x,t)$ is proposed as,

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)\frac{t^{n\alpha}}{(n\alpha)!} \quad (9)$$

To find the mathematical solutions of (8), $u_i(x,t)$ may be written in the form,

$$s_i = \sum_{n=0}^{i} u_n(x,t) = \sum_{n=0}^{i} f_n(x)\frac{t^{n\alpha}}{(n\alpha)!} \quad (10)$$

Step 4 The Elzaki residual function from (8) can be written as,

$$Res_i(x,t) = u_i(x,t) - G(x,t) - E^{-1}[v^\alpha E\{D_x^2\, u_{i-1}(x,t) + u_{i-1}(x,t)\{1-u_{i-1}(x,t)\}\}]$$
(11)

and hence the values of $f_n(x)$ may be obtained by putting $n = 0,1,2, \ldots \ldots$ in the relation,

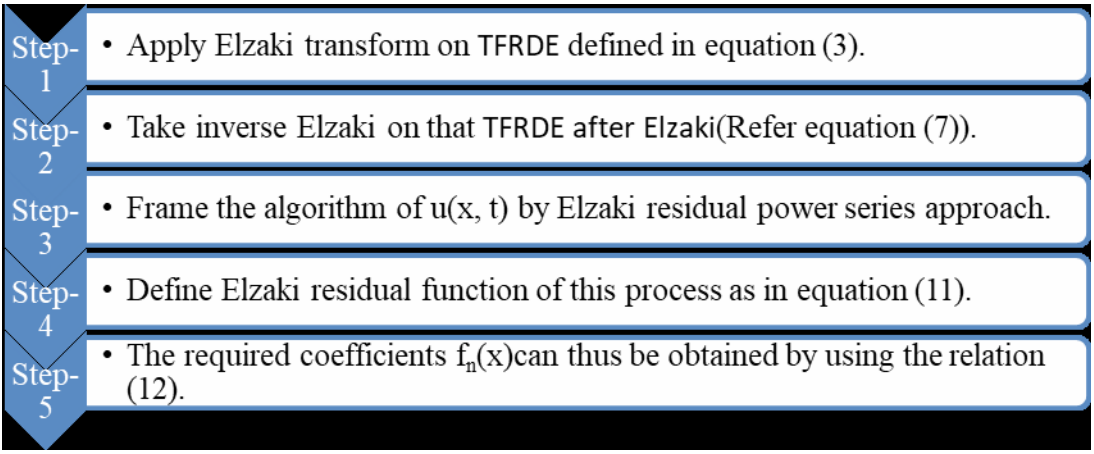
$$t^{-i\alpha} Res_i(x,t)/_{t=0} = 0 \quad (12)$$

Substituting these values of $f_n(x)$ obtained from (12) in equation (9) the approximate solution of TFRDE is obtained analytically.

The pseudo code of the methodology can be demonstrated as in **Figure 1**.

Figure 1

Pseudo code of the Methodology



Numerical Experiment

Applying Elzaki transform on equation (3) as,

$$E[D_t^\alpha u(x,t)] = E[D_x^2 u(x,t) + u(x,t)\{1-u(x,t)\}] \tag{13}$$

Applying the differentiation property of Elzaki transform,

$E[D_t^\alpha u(x,t)] = \frac{1}{v^\alpha}\{E\big(u(x,t)\big) - g(x,t)\}$ on equation (13), we get

$$\frac{1}{v^\alpha}\{E\big(u(x,t)\big) - g(x,t)\} = E[D_x^2 u(x,t) + u(x,t)\{1-u(x,t)\}]$$

$$i.e.\, E\big(u(x,t)\big) = g(x,t) + v^\alpha E[D_x^2 u(x,t) + u(x,t)\{1-u(x,t)\}] \tag{14}$$

Taking inverse Elzaki transform in equation (14), we get

$$u(x,t) = G(x,t) + E^{-1}[v^\alpha E[D_x^2 u(x,t) + u(x,t)\{1-u(x,t)\}]] \tag{15}$$

here G(x, t) is the primary circumstance (initial solution) of given equation.

By this method algorithm of $u(x,t)$ is proposed as,

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)\frac{t^{n\alpha}}{(n\alpha)!} \tag{16}$$

To find the mathematical solutions of (15), $u_i(x,t)$ may be written in the form,

$$s_i = \sum_{n=0}^{i} u_n(x,t) = \sum_{n=0}^{i} f_n(x)\frac{t^{n\alpha}}{(n\alpha)!} \tag{17}$$

The Elzaki residual function from (15) can be written as,

$$Res_i(x,t) = u_i(x,t) - G(x,t) - E^{-1}[\upsilon^\alpha E\{D_x^2 u_{i-1}(x,t) + u_{i-1}(x,t)\{1 - u_{i-1}(x,t)\}\}]$$

(18)

When $i = 0$ from equation (18),

$Res_0(x,t) = u_0(x,t) - G(x,t)$ and from equation (12),

$0 = u_0(x,t) - G(x,t)$ i.e. $u_0(x,t) = G(x,t)$

$$u_0(x,t) = u(x,0) = f_0(x) = \mu \tag{19}$$

When $i = 1$ from equation (18),

$$Res_1(x,t) = u_1(x,t) - G(x,t) - E^{-1}[\upsilon^\alpha E\{D_x^2 u_0(x,t) + u_0(x,t)\{1 - u_0(x,t)\}\}]$$

with the conditions $u_1(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\alpha!}$ then we can obtain,

$$Res_1(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\alpha!} - G(x,0) - E^{-1}[\upsilon^\alpha E\{D_x^2 u_0(x,t) + u_0(x,t)\{1 - u_0(x,t)\}\}]$$

$$= \mu + f_1(x)\frac{t^\alpha}{\alpha!} - \mu - E^{-1}[\upsilon^\alpha E\{D_x^2 u_0(x,0) + u_0(x,0)\{1 - u_0(x,0)\}\}]$$

$$= f_1(x)\frac{t^\alpha}{\alpha!} - E^{-1}[\upsilon^\alpha E\{D_x^2\mu + \mu(1-\mu)\}]$$

$$= f_1(x)\frac{t^\alpha}{\alpha!} - E^{-1}[\upsilon^\alpha E\{0 + \mu(1-\mu)\}]$$

$$= f_1(x)\frac{t^\alpha}{\alpha!} - \mu(1-\mu)E^{-1}[\upsilon^\alpha E(1)]$$

$$= f_1(x)\frac{t^\alpha}{\alpha!} - \mu(1-\mu)E^{-1}[\upsilon^{\alpha+2}]$$

$$= f_1(x)\frac{t^\alpha}{\alpha!} - \mu(1-\mu)\frac{t^\alpha}{\alpha!}$$

$$= \{f_1(x) - \mu(1-\mu)\}\frac{t^\alpha}{\alpha!}$$

Then after solving $t^{-\alpha} Res_1(x,t)_{t=0} = 0$ gives that

$$f_1(x) - \mu(1-\mu) = 0 \;\; i.e.\, f_1(x) = \mu(1-\mu) \tag{20}$$

When $i = 2$ from equation (18)

$$Res_2(x,t) = u_2(x,t) - G(x,t) - E^{-1}[\upsilon^\alpha E\{D_x^2 u_1(x,t) + u_1(x,t)\{1 - u_1(x,t)\}\}],$$

with conditions $u_1(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\alpha!}$ and

$u_2(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\alpha!} + f_2(x)\frac{t^{2\alpha}}{(2\alpha)!}$, we get

$Res_2(x,t) = f_0(x) + f_1(x)\frac{t^\alpha}{\alpha!} + f_2(x)\frac{t^{2\alpha}}{(2\alpha)!} - f_0(x) - E^{-1}[v^\alpha E\{D_x^2 u_1(x,t) + u_1(x,t)\{1 - u_1(x,t)\}\}]$

$= f_1(x)\frac{t^\alpha}{\alpha!} + f_2(x)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}\left[v^\alpha E\left\{D_x^2\left(f_0(x) + f_1(x)\frac{t^\alpha}{\alpha!}\right) + \left(f_0(x) + f_1(x)\frac{t^\alpha}{\alpha!}\right)(1 - f_0(x) - f_1(x)\frac{t^\alpha}{\alpha!})\right\}\right]$

$= f_1(x)\frac{t^\alpha}{\alpha!} + f_2(x)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}\left[v^\alpha E\left\{\left(f_0(x) - f_0^2(x) - f_0(x)f_1(x)\frac{t^\alpha}{\alpha!} + f_1(x)\frac{t^\alpha}{\alpha!} - f_0(x)f_1(x)\frac{t^\alpha}{\alpha!} - f_1^2(x)\frac{t^{2\alpha}}{(2\alpha)!}\right)\right\}\right]$

$= f_1(x)\frac{t^\alpha}{\alpha!} + f_2(x)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}[v^\alpha\{(f_0(x)v^2 - f_0^2(x)v^2 - f_0(x)f_1(x)v^{\alpha+2} + f_1(x)v^{\alpha+2} - f_0(x)f_1(x)v^{\alpha+2} - f_1^2(x)v^{2\alpha+2})\}]$

$= f_1(x)\frac{t^\alpha}{\alpha!} + f_2(x)\frac{t^{2\alpha}}{(2\alpha)!} - E^{-1}[f_0(x)v^{\alpha+2} - f_0^2(x)v^{\alpha+2} - 2f_0(x)f_1(x)v^{2\alpha+2} + f_1(x)v^{2\alpha+2} - f_1^2(x)v^{3\alpha+2}]$

$= f_1(x)\frac{t^\alpha}{\alpha!} + f_2(x)\frac{t^{2\alpha}}{(2\alpha)!} - f_0(x)\frac{t^\alpha}{\alpha!} + f_0^2(x)\frac{t^\alpha}{\alpha!} + 2f_0(x)f_1(x)\frac{t^{2\alpha}}{(2\alpha)!} - f_1(x)\frac{t^{2\alpha}}{(2\alpha)!} + f_1^2(x)\frac{t^{3\alpha}}{(3\alpha)!}$

Therefore, from $t^{-2\alpha} Res_2(x,t)/_{t=0} = 0$ we have,

$f_2(x) + 2f_0(x)f_1(x) - f_1(x) = 0 \quad i.e.\ f_2(x) = f_1(x) - 2f_0(x)f_1(x)$

$i.e.\ f_2(x) = \mu(1-\mu)(1-2\mu)$ (21)

Now, the second approximate solution is,

$u_2(x,t) = \mu + \mu(1-\mu)\frac{t^\alpha}{\alpha!} + \mu(1-\mu)(1-2\mu)\frac{t^{2\alpha}}{(2\alpha)!}$ (22)

Similarly, the n^{th} coefficient of $u(x,t)$ is $f_n(x) = \mu(1-\mu)(1-2\mu)\ldots\{1-(n-1)\mu\}$

.

$u_n(x,t) = \mu + \mu(1-\mu)\frac{t^\alpha}{\alpha!} + \mu(1-\mu)(1-2\mu)\frac{t^{2\alpha}}{(2\alpha)!} + \cdots + \mu(1-\mu)(1-2\mu)\ldots\{1-(n-1)\mu\}\frac{t^{n\alpha}}{(n\alpha)!}$

(23)

Results and Discussion

The approximate numerical solution of the above equation for different values of t is calculated and is compared with exact solution for different number of terms. The graph of exact solution of this equation and its approximate numerical solutions with different number of terms are shown in Figure 2, Figure 3 and Figure 4 with values of alpha = 1.0, 0.5 and 0.25 respectively. The errors at different time levels are shown in Figure 5 which shows that the solution obtained by this method is reliable and efficient.

Figure 2

Both solutions when alpha=1

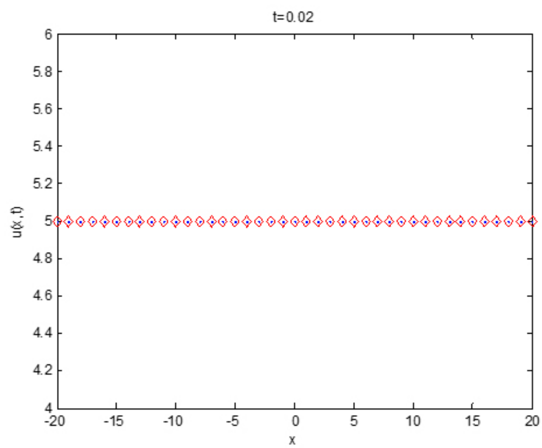


Figure 3

Both solutions when alpha=0.5

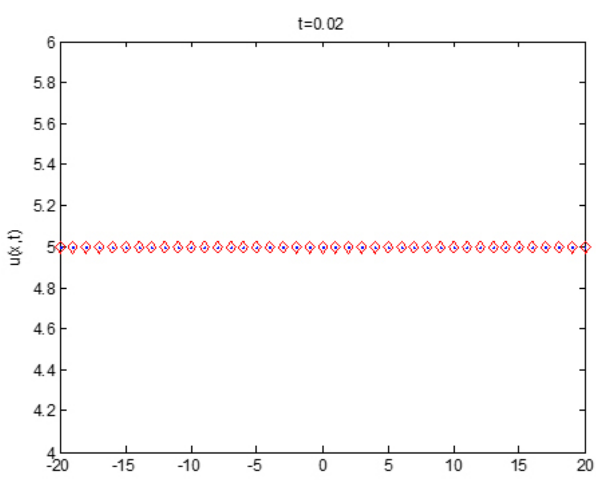


Figure 4

Both solutions when alpha=0.25

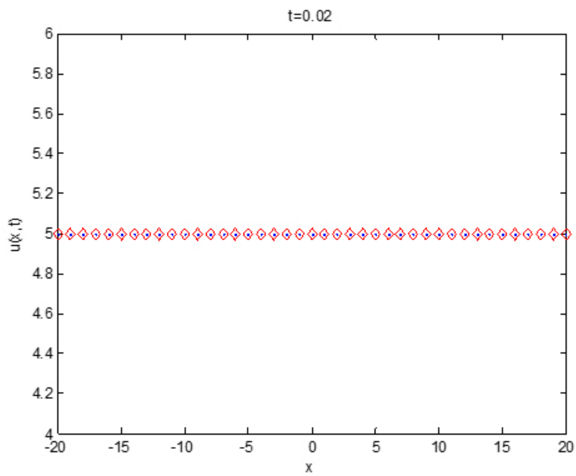
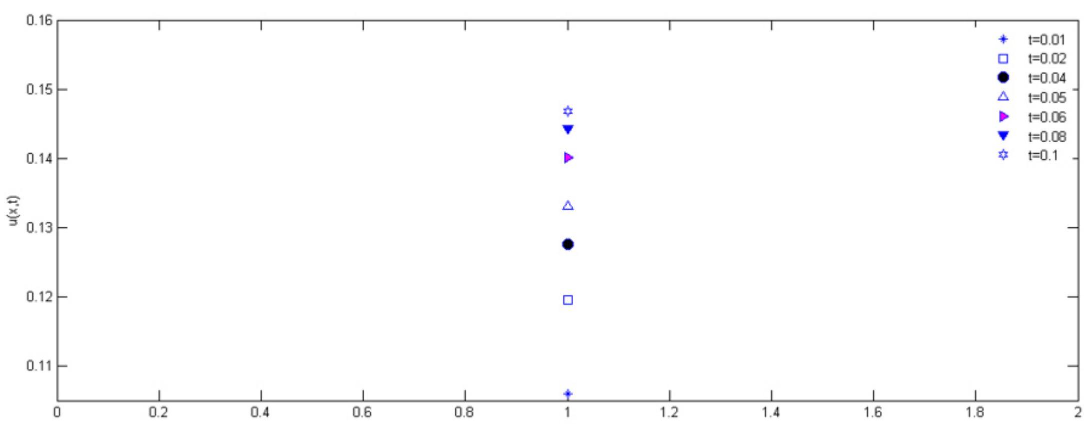


Figure 5

Errors at different time levels



Conclusion

In current paper, a new and reliable approach is constructed to find the approximate solutions of TFRDE by using ERPSM. This approach combines the Elzaki transform and residual power series method that is the enhancement of classical residual power series method. The advantage of this approach is to decrease the computational work with less error for finding the solution in residual power series form. The coefficients of this power series solution are determined in the above successive steps. The considered equation is solved by above approach that verified its ability to solve such equation with sufficient correctness and reliable computation steps. This new method is beneficial for solution of TFRDE. This method also provides simple as well as accurate algorithms for finding approximate solutions of this equation.

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