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Numerical Solutions of Non-Linear Systems of ODES: Lotka–Volterra Predator–Prey Model

NARAYAN SAPKOTA[1,2*], KHIM B. KHATTRI [2]

ABSTRACT. This study investigates the dynamic behavior of predator–prey interactions using the classical Lotka–Volterra system of nonlinear ordinary differential equations. We employ both analytical and numerical techniques to analyze population fluctuations of hares and lynx, based on historical Hudson Bay Company records. An approximate analytical solution is derived using the Ritz method, while a numerical solution is obtained via the fourth-order Runge–Kutta (RK4) method. To enhance model realism, we estimate system parameters $(\alpha, \beta, \gamma, \delta)$ using Simulated Annealing (SA), a global optimization technique well-suited for non-convex landscapes. The model is calibrated against empirical data, and SA-based optimization achieved mean RMSE values of 4.12 for hares and 4.01 for lynx across five independent runs. Comparative plots between observed and predicted populations confirm the model's ability to capture oscillatory behavior and phase shifts. This work compares analytical and numerical solution methods for the Lotka–Volterra system using parameters estimated via Simulated Annealing, demonstrating the relative strengths of the Ritz and Runge–Kutta approaches in modeling real-world population dynamics.

Keywords: Lotka–Volterra Model, Nonlinear Ordinary Differential Equations, Ritz method, Runge-Kutta method, Simulated Annealing.

[1]Department of Computer Engineering, Everest Engineering College, Nepal
E-mail: narayan_sapkota@outlook.com
[2]Department of Mathematics, School of Science, Kathmandu University, Nepal
E-mail: khimkhattri@ku.edu.np
[*] Corresponding author
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1. Introduction

The Lotka–Volterra predator–prey model stands as a foundational framework in the study of nonlinear dynamical systems, originally developed to describe ecological interactions between predator and prey populations [18, 27]. This model captures the cyclical nature of population dynamics through a system of first-order nonlinear differential equations, assuming that prey populations grow exponentially in the absence of predators, while predator populations decline without prey. Their interaction introduces a regulatory feedback mechanism that leads to oscillatory behavior in both species [12]. Due to its simplicity and interpretability, the Lotka–Volterra model has been extensively studied and applied across a wide range of disciplines. In theoretical ecology, it offers insights into species interactions such as wolves and rabbits in terrestrial ecosystems or bass and redear fish in aquatic environments [7]. It has also been used in conservation biology to simulate intervention scenarios, including predator control and endangered species management.

Beyond ecology, the model has found applications in fields such as epidemiology, where it helps simulate interactions between hosts and vectors in diseases like malaria and dengue fever [2, 11]. In economics, it has been adapted to describe competitive market dynamics where firms compete for market dominance, mimicking predator–prey relationships in terms of resource or customer share [19, 28]. The model also extends to engineering systems such as power grids and communication networks, where it describes the balance between supply and demand or the flow of competing data packets [25, 3, 26]. In cell biology and immunology, predator–prey analogues represent virus-infected cells and immune responses, contributing to the understanding of infection dynamics and therapeutic design [10, 21]. Social sciences and artificial intelligence have further leveraged Lotka–Volterra frameworks to simulate cultural interactions, ideological shifts, and the dynamics of neural networks [17, 22]. Despite its versatility, the classical Lotka–Volterra model presents analytical challenges due to its inherent nonlinearity. While qualitative analysis using phase plane methods and stability theory provides insight into equilibrium behavior [1], closed-form solutions are rarely available, especially under realistic ecological conditions involving stochasticity, migration, or environmental constraints [11]. Consequently, numerical approximation methods such as Euler's method and Runge–Kutta schemes have become essential tools for exploring these systems [13].

This study focuses on evaluating the Lotka–Volterra system using both analytical and numerical strategies. We begin with an approximate analytical solution via the Ritz method, highlighting its utility and limitations in capturing the system's inherent oscillations [29]. We then implement a fourth-order Runge–Kutta (RK4) method to numerically simulate population dynamics over time. To align the model with real-world observations, we perform parameter estimation using Simulated Annealing—a global optimization algorithm well-suited for nonlinear systems with complex error surfaces. The model is calibrated against historical lynx–hare data from the Hudson Bay Company, and predictive accuracy is evaluated using root mean square error (RMSE). Finally, we analyze the system's long-term behavior through phase portraits and discuss the broader applicability of the approach in ecological modeling and interdisciplinary research.

2. Lotka–Volterra Model

2.1. **Mathematical Formulation.** The Lotka–Volterra predator-prey model [7, 1, 29] is governed by the following system of two nonlinear ODEs:

$$\begin{cases} \frac{dx}{dt} = \alpha x(t) - \beta x(t)y(t) \\ \frac{dy}{dt} = \delta x(t)y(t) - \gamma y(t) \end{cases} \tag{1}$$

where:

- $x(t)$ represents the prey population at time t,
- $y(t)$ represents the predator population at time t,
- α is the growth rate of the prey in the absence of predators,
- β is the predation rate, describing how often predators catch prey,
- δ is the rate at which predators reproduce based on the availability of prey,
- γ is the death rate of the predators in the absence of food (prey).

The constants α, β, δ and γ are all positive. When the predator is absent, the prey population grows at a rate proportional to its current population. This is described by the equation $\frac{dx}{dt} = \alpha x(t)$ and $y(t) = 0$. Similarly, in the absence of the prey, the predator population declines, with the rate of change given by $\frac{dy}{dt} = -\gamma y(t)$ and $x(t) = 0$.

Experimental studies have documented numerous prey-predator interactions that closely resemble the dynamics predicted by the Lotka–Volterra model. These investigations have examined different species pairs, including the Canada lynx and snowshoe hare [14], as well as moose and wolf populations in Isle Royale National Park [16] and more examples can be found on [29].

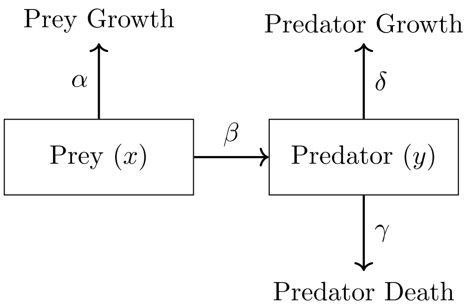

FIGURE 1. Schematic representation of the Lotka–Volterra predator–prey model showing growth, interaction, and death rates.

2.1.1. *Biological Interpretation and Model Assumptions.* The prey are assumed to have an unlimited food supply and reproduce exponentially, except when preyed upon. This exponential growth is represented in Equation (1) by the term αx. The rate of predation on the prey is assumed to be proportional to the rate at which predators and prey encounter one another, represented by βxy. If either x or y is zero, no predation occurs. Hence, the prey equation in Equation (1) describes the rate of change of the prey population as its growth rate minus the predation rate. The term δxy represents the growth of the predator population, which is proportional to the number of prey consumed. The predator growth rate differs from the predation rate, which is why a different constant is used. The term γy represents the predator mortality rate, encompassing natural death or emigration. In the absence of prey, this leads to exponential decay. Therefore, the predator equation in Equation (1) expresses the rate of change of the predator population as the predation rate minus its intrinsic death rate.

The Lotka–Volterra predator-prey model makes several assumptions about the environment and the biology of the populations [7], [29], [6]:

- The prey population has an unlimited food supply at all times.
- The predator population's food supply depends entirely on the prey population.
- The rate of population change is proportional to the population size.
- Environmental factors do not favor one species over the other, and genetic adaptation does not play a significant role during the modeling period.
- Predators have an unlimited appetite.
- Both populations are modeled as a single variable, implying that spatial or age distribution does not influence population dynamics.

2.1.2. *Significance of the Model in Real-World Applications.* The Lotka–Volterra model serves as a foundational framework in ecology, particularly for understanding the dynamics between interacting species in an ecosystem. It illustrates how changes in the population of one species affect the other, showing the cyclical nature of predator-prey dynamics: increases in prey populations lead to higher predator populations, which then reduce the prey population, causing a subsequent decline in predators. In ecology, the model is commonly applied to species such as predators (e.g., wolves) and prey (e.g., rabbits). It can also be used to simulate various scenarios, such as conservation efforts or managing endangered species. The model can be extended to include factors like environmental carrying capacity, external predation pressures, migration, or disease, all of which are nonlinear and complicate the model's analytical solutions.

Beyond ecology, the Lotka-Volterra model has been extensively adapted to describe a wide range of systems across various disciplines. In epidemiology, the model is used to simulate interactions between disease-carrying organisms, such as mosquitoes, and the spread of infections. A prominent application is in the study of vector-borne diseases, where interactions between mosquito populations (the vectors) and human populations (the hosts) are crucial for understanding disease dynamics, such as malaria and dengue fever [2]. This application is key for predicting disease outbreaks and devising control strategies [11]. In economics, the model has been adapted to represent competitive market dynamics [28]. Specifically, it has been applied to model competition between firms, where one firm acts as the "predator" and the other as the "prey" in terms of market share [19]. Additionally, the model has been used to study resource allocation in various economic sectors, providing insights into market stability and the long-term behavior of competing entities [23].

In cell biology, the Lotka-Volterra model helps represent interactions between different cell types, such as infected and uninfected cells, to understand infection dynamics within populations [10]. It has been used to simulate viral infections, examining interactions between virus-infected cells and immune cells [21] [20]. These insights are crucial for developing effective treatments, vaccines, and for understanding how infections spread and persist within populations. In engineering, the Lotka-Volterra model is employed to simulate systems with two interacting components, such as the balance between supply and demand or resource allocation in networks. For example, in power grid management, the model describes the interactions between energy supply and consumption [25] [26]. Similarly, in communication networks, it has been used to represent traffic flow, where competing entities (such as data packets) interact and influence overall network performance [3] [4] [5]. These applications illustrate how predator-prey dynamics govern systems in which competition or cooperation between components drives the overall behavior of the system.

Finally, the Lotka-Volterra model can also be applied to understand social behavior dynamics, such as interactions between competing social groups, ideologies, or cultures [17]. It has been adapted for use in neural networks to study the dynamics of interacting components in complex systems [22]. This broad applicability across various fields highlights the model's versatility in describing complex interactions beyond its original ecological context.

3. Analytical Methods

Despite the increasing applications of the Lotka-Volterra model, no closed-form analytical solution for this predator-prey system has been established in the literature [24]. The Equation (1) is recognized as conservative, meaning that its solutions must exhibit periodic behavior and do not have a simple expression in terms of the usual trigonometric functions; however, the exact analytical expressions for these solutions remain unknown [24]. In this section, we will first discuss the theoretical background of the Ritz method also known as principle of harmonic balance, followed by its application to the LV system, as demonstrated in [8].

3.1. **Theoretical Background of Ritz Method.** The Ritz method provides an approximate solution, $x(t)$, to the equation

$$\ddot{x} + f(x) = 0$$

by ensuring that the functional

$$J = \int_{t_a}^{t_b} F(x, \dot{x}, t)\, dt$$

is minimized. Here, $F(x, \dot{x}, t)$ is selected such that the solution to the minimum of J, governed by the Euler-Lagrange equation, corresponds to the differential equation we aim to solve:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = \ddot{x} + f(x) = E(x) = 0.$$

The approximate solution $x(t)$ is expressed as

$$x(t) = \sum a_i \varphi_i(t)$$

where the $\varphi_i(t)$ are a linearly independent set chosen based on prior knowledge of the differential equation's behavior.

To determine $\tilde{x}(t)$, the functional J must be minimized with respect to the n coefficients a_i. If we impose the condition that $\varphi_i(t_a) = \varphi_i(t_b) = 0$, or that φ_i is periodic over the interval $t_a - t_b$, we obtain the following conditions:

$$\int_{t_a}^{t_b} \varphi_i(t) E(\tilde{x}(t))\, dt = 0, \quad i = 1, \ldots, n$$

For oscillatory systems, the Ritz method simplifies the process by eliminating the need to perform the n integrations mentioned earlier. When we take $\tilde{x}(t) = A\cos(\omega t)$ as an approximate solution to $\ddot{x} + f(x) = 0$, we find, through evaluating the integral, that the Ritz method is equivalent to selecting ω and A such that $\tilde{x}(t)$ satisfies the differential equation, while disregarding the higher harmonics produced by $f(\tilde{x})$. In other words, the sum of the coefficients of $\cos(\omega t)$ in $E(\tilde{x}(t))$ must be zero. Similarly, when we take $\tilde{x}(t) = \sum_{n=0}^{N} a_n \cos(n\omega t)$, we find that the coefficients of $\cos(n\omega t)$, for $n = 0, \ldots, N$ in $E(\tilde{x}(t))$ must balance to zero, which is the essence of the principle of harmonic balance.

3.2. **Ritz Method for Solving the LV System.** Equilibrium points, also known as fixed points, are locations where both populations remain constant over time, meaning $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. To determine the equilibrium points, set the right-hand sides of the equation (1) equal to zero:

$$\alpha x - \beta xy = 0 \tag{2}$$

$$\delta xy - \gamma y = 0 \tag{3}$$

The Equation (2), has two possibilities:

(1) $x = 0$ (which is a trivial equilibrium corresponding to the extinction of the prey population),
(2) $\alpha - \beta y = 0 \Rightarrow y = \frac{\alpha}{\beta}$ (for a non-zero prey population).

Similarly, Equation (3) has two possibilities

(1) $y = 0$ (which corresponds to the extinction of the predator population),
(2) $\delta x - \gamma = 0 \Rightarrow x = \frac{\gamma}{\delta}$ (for a non-zero predator population).

Thus, the only non-trivial equilibrium point is:

$$x^* = \frac{\gamma}{\delta}, \quad y^* = \frac{\alpha}{\beta}$$

As a result, the system (2) has two singularities at the points $(x, y) = (0, 0)$ and $\left(\frac{y}{\delta}, \frac{\alpha}{\beta}\right)$. These are referred to as the *prey and predator equilibrium populations*. By linearizing around these singularities, we find that the first is a saddle point, while the second is a center. We will apply the equivalent Ritz method to derive approximate solutions in the vicinity of the center. We postulate an approximate solution of the form:

$$\begin{aligned} x(t) &= A + B\cos(\omega t) + C\sin(\omega t) \\ y(t) &= D + E\cos(\omega t) + F\sin(\omega t) \end{aligned} \tag{4}$$

Differentiating with respect to t :

$$\begin{aligned} \frac{dx}{dt} &= -B\omega\sin(\omega t) + C\omega\cos(\omega t) \\ \frac{dy}{dt} &= -E\omega\sin(\omega t) + F\omega\cos(\omega t) \end{aligned} \tag{5}$$

Substituting these solutions into the differential equations (1), we get

$$\frac{dx}{dt} = \alpha\left(A + B\cos(\omega t) + C\sin(\omega t)\right)$$
$$-\beta\left(A + B\cos(\omega t) + C\sin(\omega t)\right)\left(D + E\cos(\omega t) + F\sin(\omega t)\right)$$

$$-B\omega\sin(\omega t) + C\omega\cos(\omega t) = \alpha\left(A + B\cos(\omega t) + C\sin(\omega t)\right)$$
$$-\beta\left(A + B\cos(\omega t) + C\sin(\omega t)\right)\left(D + E\cos(\omega t) + F\sin(\omega t)\right)$$

Similarly,

$$\frac{dy}{dt} = \delta\left(A + B\cos(\omega t) + C\sin(\omega t)\right)$$
$$\left(D + E\cos(\omega t) + F\sin(\omega t)\right) - \gamma\left(D + E\cos(\omega t) + F\sin(\omega t)\right)$$

$$-E\omega\sin(\omega t) + F\omega\cos(\omega t) = \delta\left(A + B\cos(\omega t) + C\sin(\omega t)\right)$$
$$\left(D + E\cos(\omega t) + F\sin(\omega t)\right) - \gamma\left(D + E\cos(\omega t) + F\sin(\omega t)\right)$$

By simplifying above equations and setting the coefficients of $\cos(\omega t)$, $\sin(\omega t)$, and the constant terms to zero, we derive a system of algebraic equations for the variables A, B, C, D, E, F, and ω as:

$$\begin{aligned}
\alpha C - \beta AF - \beta CD + \omega B &= 0\\
\omega C - \alpha B + \beta AE + \beta BD &= 0\\
\alpha A - \beta AD - \left(\frac{\beta BE}{2}\right) - \left(\frac{\beta CF}{2}\right) &= 0\\
\omega E - \gamma F + \delta AF + \delta CD &= 0\\
\omega F + \gamma E - \delta AE - \delta BD &= 0\\
\delta AD - \gamma D + \left(\frac{\delta BE}{2}\right) + \left(\frac{\delta CF}{2}\right) &= 0
\end{aligned}$$

From the initial condition($t = 0$), we have:

$$\begin{aligned}
x(0) &= A + B\\
y(0) &= D + E
\end{aligned}$$

It is clear that for the desired equilibrium point to be achieved, we must set $A = \frac{\gamma}{\delta}$ and $D = \frac{\alpha}{\beta}$. By solving the system of equations above, we derive the following results:

$$\begin{aligned}
B &= x(0) - \frac{\gamma}{\delta}\\
C &= \pm\left(y(0) - \frac{\alpha}{\beta}\right)\left(\frac{\beta}{\delta}\right)\sqrt{\frac{\gamma}{\alpha}}\\
E &= y(0) - \frac{\alpha}{\beta}\\
F &= \pm(x(0) - \frac{\gamma}{\delta})\left(\frac{\delta}{\beta}\right)\sqrt{\frac{\alpha}{\gamma}}\\
\omega &= \pm\sqrt{\gamma\alpha}
\end{aligned} \tag{6}$$

Thus, the approximate solutions are given by

$$\begin{aligned}
x(t) &= \frac{\gamma}{\delta} + (x(0) - \frac{\gamma}{\delta})\cos(\sqrt{\gamma\alpha}t) - (y(0) - \frac{\alpha}{\beta})\left(\frac{\beta}{\delta}\right)\sqrt{\frac{\gamma}{\alpha}}\sin(\sqrt{\gamma\alpha}t)\\
y(t) &= \frac{\alpha}{\beta} + (y(0) - \frac{\alpha}{\beta})\cos(\sqrt{\gamma\alpha}t) + (x(0) - \frac{\gamma}{\delta})\left(\frac{\delta}{\beta}\right)\sqrt{\frac{\alpha}{\gamma}}\sin(\sqrt{\gamma\alpha}t)
\end{aligned} \tag{7}$$

These equations describe an elliptical trajectory in the x-y plane with closed orbits around the equilibrium point.This indicates that the populations of both species oscillate in a cyclical manner, as predators and prey undergo regular cycles of growth and decline.

3.3. **Limitations of the Ritz Method.** While the Ritz method provides an efficient way to approximate periodic or quasi-periodic behavior in nonlinear systems, it is subject to several important limitations:

- The method assumes that the solution remains smooth and approximately harmonic. It performs well near equilibrium but fails under strongly nonlinear or chaotic conditions.
- Discontinuities, sharp transitions, or external forcing (e.g., environmental perturbations) are not handled well due to the fixed functional form.

- The quality of the solution heavily depends on the choice of trial functions. Inadequate basis functions can lead to poor approximations even if residuals are minimized.

Despite these limitations, the Ritz method remains valuable for gaining qualitative insight into the dynamics of ecological models and for validating trends observed in numerical simulations.

3.4. **Stability Analysis.** To determine the stability of the equilibrium point, we linearize the system near the equilibrium using the Jacobian matrix. The Jacobian matrix is given by:

$$J = \begin{pmatrix} \frac{\partial}{\partial x}(\alpha x - \beta xy) & \frac{\partial}{\partial y}(\alpha x - \beta xy) \\ \frac{\partial}{\partial x}(\delta xy - \gamma y) & \frac{\partial}{\partial y}(\delta xy - \gamma y) \end{pmatrix} = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix}$$

Jacobian at the equilibrium point $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$:

$$J = \begin{pmatrix} \alpha - \beta\frac{\alpha}{\beta} & -\beta\frac{\gamma}{\delta} \\ \delta\frac{\alpha}{\beta} & \delta\frac{\gamma}{\delta} - \gamma \end{pmatrix} = \begin{pmatrix} 0 & -\beta\frac{\gamma}{\delta} \\ \delta\frac{\alpha}{\beta} & 0 \end{pmatrix}$$

The eigenvalues λ of the Jacobian matrix can be found by solving the characteristic equation:

$$\det(J - \lambda I) = 0$$

This simplifies to:

$$\lambda^2 = \beta\gamma\frac{\alpha}{\delta}$$

Thus, the eigenvalues are:

$$\lambda = \pm\sqrt{\beta\gamma\frac{\alpha}{\delta}}$$

Based on these eigenvalues, we can conclude the stability of the equilibrium points. If the real parts of the eigenvalues are positive, the equilibrium point is unstable, while if the real parts are negative, the equilibrium point is stable. The sign of the eigenvalues determines whether the system experiences growth or decay around the equilibrium point, providing insight into the system's behavior over time.

4. Numerical Methods

4.1. **Runge-Kutta Methods.** The Runge-Kutta family of methods is one of the most commonly used numerical techniques for solving ODEs. The most widely known is the fourth-order Runge-Kutta method (RK4), which provides a good balance between computational efficiency and accuracy. The RK4 method is applied as follows to solve Equation (1):

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{h}{6}(L_1 + 2L_2 + 2L_3 + L_4) \tag{8}$$

where h is the step size and the terms L_1, L_2, L_3, L_4 are defined as:

$$\begin{aligned} L_1 &= f(t_n, \mathbf{x}_n) \\ L_2 &= f\left(t_n + \frac{h}{2}, \mathbf{x}_n + \frac{h}{2}L_1\right) \\ L_3 &= f\left(t_n + \frac{h}{2}, \mathbf{x}_n + \frac{h}{2}L_2\right) \\ L_4 &= f(t_n + h, \mathbf{x}_n + hL_3) \end{aligned}$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \tag{9}$$

where h is the step size and the terms k_1, k_2, k_3, k_4 are defined as:

$$k_1 = f(t_n, \mathbf{y}_n)$$
$$k_2 = f\left(t_n + \frac{h}{2}, \mathbf{y}_n + \frac{h}{2}k_1\right)$$
$$k_3 = f\left(t_n + \frac{h}{2}, \mathbf{y}_n + \frac{h}{2}k_2\right)$$
$$k_4 = f(t_n + h, \mathbf{y}_n + hk_3)$$

In this method, the values k_1, k_2, k_3, k_4 and L_1, L_2, L_3, L_4 represent different approximations of the slope at various points within the interval, and their weighted average provides the next value in the solution.

5. Results and Discussion

5.1. **Experimental Setup.** The table below, Table 1, illustrates a sample dataset of lynx (predator) and hare (prey) populations, measured in thousands of individuals, recorded by the Hudson Bay Company from 1900 to 1935 in a forest in northern Canada [14], [9], [15]. The initial populations of the predator and prey, taken from the year 1900, were $x(0) = 12.82$ and $y(0) = 7.13$. The cyclical pattern (see Figure 2)in the data suggests periodic fluctuations in both species, which align with the theoretical predictions of the Lotka–Volterra equations. The observed trend indicates that as hare populations increase, lynx populations follow with a time lag, leading to periodic oscillations.

Table 1. Samples of Lynx and Hares in thousands of individuals from year 1900 to 1935 [15]

Year	**Hares**	**Lynxes**	**Year**	**Hares**	**Lynxes**
1900	12.82	7.13	1918	4.50	6.82
1901	4.72	9.47	1919	11.21	3.19
1902	4.73	14.86	1920	56.60	3.52
1903	37.22	31.47	1921	69.63	9.94
1904	69.72	60.57	1922	77.74	20.30
1905	57.78	63.51	1923	80.53	31.99
1906	28.68	54.70	1924	73.38	42.36
1907	23.37	6.30	1925	36.93	49.08
1908	21.54	3.41	1926	4.64	53.99
1909	26.34	5.44	1927	2.54	52.25
1910	53.10	11.65	1928	1.80	37.70
1911	68.48	20.35	1929	2.39	19.14
1912	75.58	32.88	1930	4.23	6.98
1913	57.92	39.55	1931	19.52	8.31
1914	40.97	43.36	1932	82.11	16.01
1915	24.95	40.83	1933	89.76	24.82
1916	12.59	30.36	1934	81.66	29.70
1917	4.97	17.18	1935	15.76	35.40

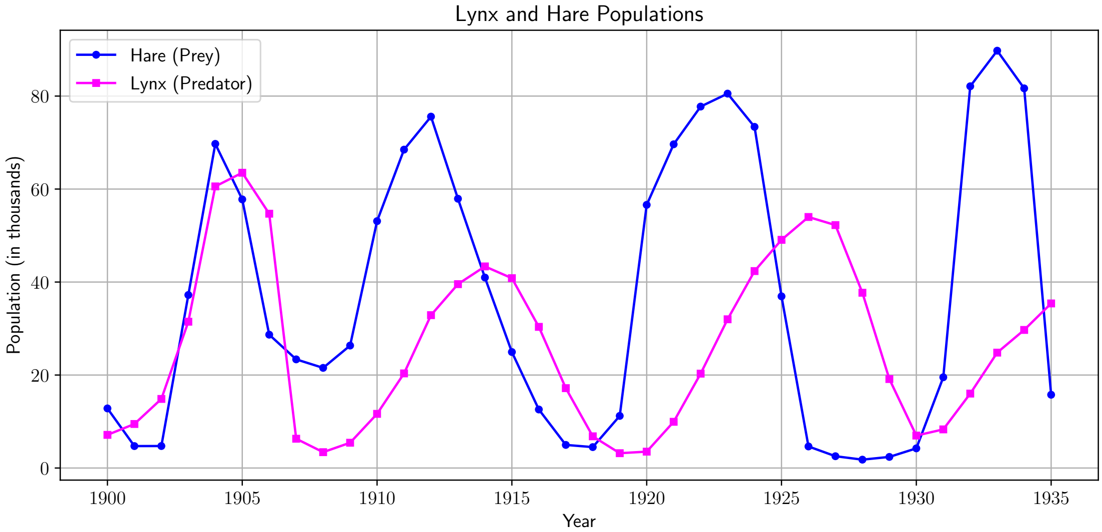

FIGURE 2. Original Data

Figure 2 illustrates the fluctuations in hare and lynx populations over time. The hare population (prey) exhibits a peak approximately every 10 years, followed by an increase in lynx population (predators). This phase shift is characteristic of predator-prey dynamics, where predator populations lag behind prey populations.

5.2. **Parameter Estimation via Simulated Annealing.** To calibrate the parameters of the Lotka–Volterra predator–prey model using empirical data, we employed the Simulated Annealing (SA) optimization algorithm. SA is a stochastic global search technique inspired by the annealing process in metallurgy. It is particularly effective for non-convex optimization landscapes, making it suitable for nonlinear systems such as the Lotka–Volterra equations.

5.2.1. *Objective Function.* Let x_i^{obs} and y_i^{obs} denote the observed hare and lynx populations at discrete time points t_i, and x_i^{sim}, y_i^{sim} the simulated populations from the model. The cost function is defined as the sum of squared residuals:

$$J(\alpha, \beta, \gamma, \delta) = \sum_{i=1}^{N} \left[(x_i^{\text{sim}} - x_i^{\text{obs}})^2 + (y_i^{\text{sim}} - y_i^{\text{obs}})^2 \right], \tag{10}$$

which is minimized using SA to obtain the best-fitting parameters.
We used the `dual_annealing` optimizer from the SciPy library to minimize the objective function. The optimization was constrained within biologically plausible bounds:

$$\alpha \in [0.1, 1.0], \quad \beta \in [0.001, 0.05], \quad \gamma \in [0.1, 2.0], \quad \delta \in [0.001, 0.05].$$

To assess robustness and repeatability, we repeated the SA optimization over five independent runs with different random seeds. For each run, the estimated parameters and RMSEs were recorded. RMSE was computed for each species as:

$$\text{RMSE}_x = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i^{\text{sim}} - x_i^{\text{obs}})^2}, \qquad \text{RMSE}_y = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (y_i^{\text{sim}} - y_i^{\text{obs}})^2}. \tag{11}$$

The average parameter values and RMSEs over five runs were:

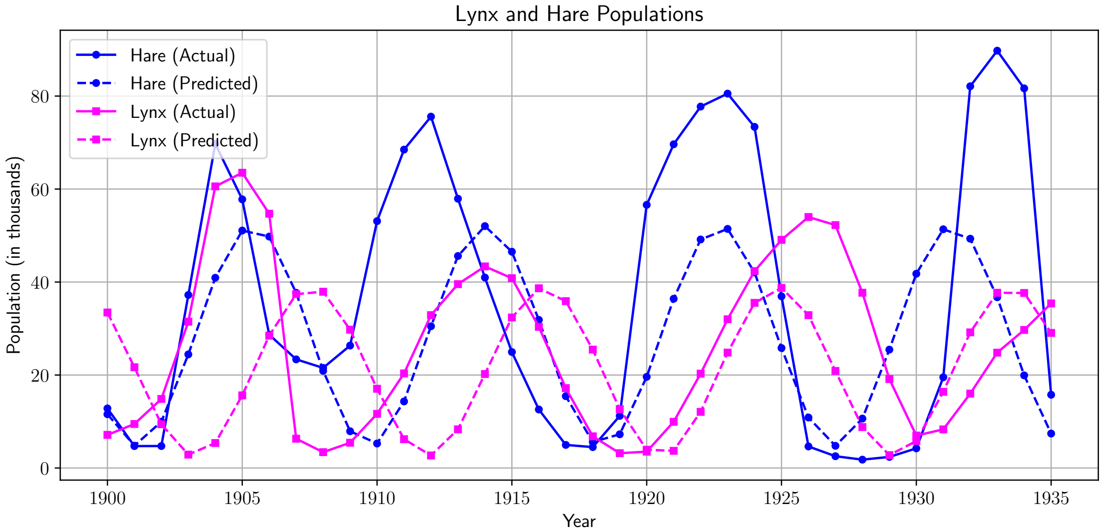

FIGURE 3. Numerical solution of a trajectory using the Ritz Method

Parameter	Mean	Std. Dev.
α	0.6923	0.0684
β	0.0334	0.0031
γ	0.7627	0.0861
δ	0.0268	0.0034

TABLE 2. Simulated Annealing estimated parameter statistics (5 runs)

By using the parameters found for α, β, γ, and δ, we have:

$$\begin{cases} \frac{dx}{dt} = 0.6923x(t) - 0.0333x(t)y(t) \\ \frac{dy}{dt} = 0.02684x(t)y(t) - 0.7627y(t) \end{cases} \tag{12}$$

The relatively small standard deviations across all parameters indicate that the SA algorithm produces stable and consistent parameter estimates.

5.3. **Ritz Method.** Figure 3 illustrates the numerical solution of the Lotka–Volterra system obtained using the Ritz method, a physics-based numerical approach. The numerical predictions closely follow the observed data, capturing the cyclic nature of the population dynamics. The model effectively reproduces the observed trends, demonstrating the validity of the chosen numerical approach for solving non-linear ODEs. Despite the general agreement, discrepancies exist between the predicted and actual values, particularly in the amplitude of population peaks. These differences may stem from simplifying assumptions in the classical Lotka–Volterra equations, such as constant interaction coefficients and absence of external environmental influences. In reality, stochastic factors like climate variations, food availability, and disease outbreaks could contribute to deviations from the idealized model.

5.4. **RK 4th Order.** The results of the LV systems simulation using the Runge-Kutta 4th-order (RK4) method demonstrate (see Figure 4) the cyclical nature of predator-prey

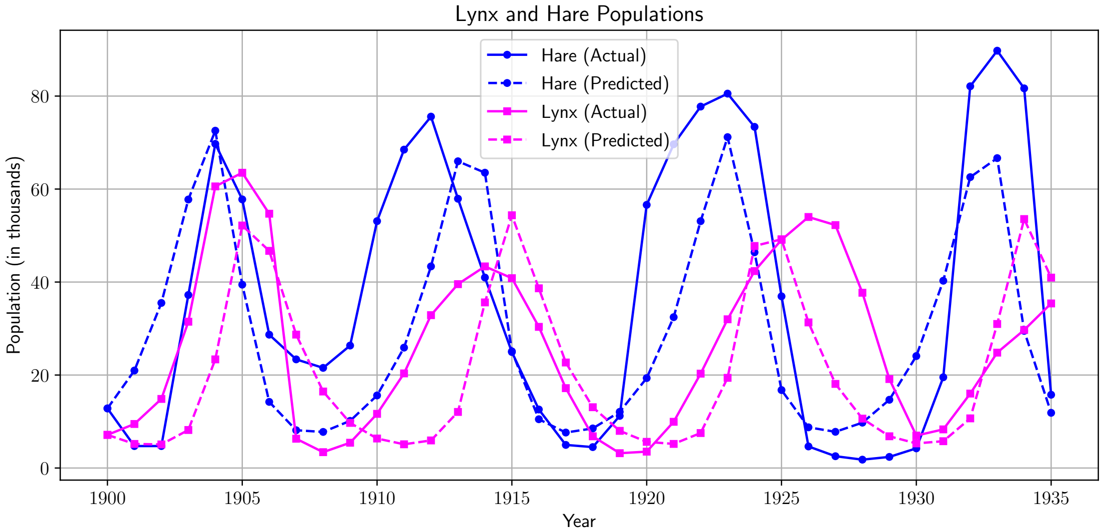

FIGURE 4. Numerical solution of a trajectory using the RK4 Method

interactions. The figure compares the actual population data of lynx and hares with the numerically predicted values over a series of years. The RK4 method effectively captures the oscillatory behavior of both species, where the prey population (hares) increases first, followed by a delayed rise in the predator population (lynx), after which both decline in a continuous cycle. The predicted results align well with the actual data, indicating that the RK4 method provides an accurate numerical approximation of the nonlinear LV equations.

While the general trends are well-represented, minor deviations exist, likely due to environmental factors not accounted for in the classical model, parameter estimation uncertainties, or natural fluctuations in the real-world data. Despite these discrepancies, the RK4 method produces smooth and stable numerical solutions, demonstrating its reliability in modeling complex ecological interactions. These results reinforce the importance of numerical methods for solving nonlinear ODEs, especially when analytical solutions are impractical. The presence of multiple trajectories corresponding to different initial predator population values suggests that the system's behavior depends on initial conditions but follows a stable, repeating pattern. The variation in initial predator populations (e.g., y = 4.0, y = 41.7, y = 7.4, etc.) affects the trajectory, but all orbits remain closed, reinforcing the theoretical prediction that predator-prey interactions remain bounded in a limit cycle rather than leading to extinction. To conclude, the phase-space plot confirms the characteristic oscillatory behavior of the LV model, demonstrating that populations follow periodic cycles rather than reaching a steady equilibrium. The RK4 method effectively captures this dynamic, providing accurate numerical approximations of the nonlinear system.

5.5. **RMSE Comparison Between RK4 and Ritz Methods.** To evaluate the accuracy of the numerical methods used to model the Hare and Lynx population dynamics, we computed the Root Mean Square Error (RMSE) for both species using the RK4 (Runge-Kutta 4th order) and Ritz methods. The RMSE quantifies the difference between the observed data and the numerical simulation results. Table 3 summarizes the RMSE values for both populations:

Method	**RMSE (Hare)**	**RMSE (Lynx)**
RK4 Method	21.77	15.54
Ritz Method	27.20	21.98

Table 3. Comparison of RMSE for Hare and Lynx Populations Using RK4 and Ritz Methods

As shown in the table, the RK4 method yields lower RMSE values for both the hare and lynx populations compared to the Ritz method. Specifically, the RMSE for the hare population is 21.77 using RK4 and 27.20 using Ritz, while for the lynx population, the RMSE is 15.54 for RK4 and 21.98 for Ritz.
This indicates that the RK4 method provides a more accurate numerical approximation of the observed population data. The higher errors in the Ritz method may be attributed to its weaker handling of rapid nonlinear dynamics present in predator-prey systems. Therefore, for problems involving sharp oscillations or highly dynamic systems like the Lotka-Volterra model, the RK4 method appears to be more suitable.

6. Conclusion and Future Research

The numerical analysis of the LV predator-prey model using the RK4 and Ritz methods demonstrates the effectiveness of numerical techniques in solving nonlinear differential equations. The results show that both methods successfully capture the oscillatory behavior of predator and prey populations, as confirmed by the time-series and phase-space plots. The RK4 method provides a stable and accurate numerical solution, while the Ritz method offers an alternative approach with comparable results. The phase plots reveal the characteristic limit cycles of predator-prey interactions, reinforcing the theoretical predictions of the LV model. However, due to inaccurate parameter estimations of α, β, δ and γ using the maximum likelihood method, the model can sometimes deviate from real-world data. While the general predator-prey cycles remain intact, small errors in parameter estimation can lead to discrepancies in amplitude and phase shifts between predicted and actual population trends.

Future research can focus on improving parameter estimation techniques to enhance the model's predictive accuracy. AI-based approaches, such as machine learning and deep learning, could be integrated to optimize the estimation of system parameters (α, β, δ and γ) reducing uncertainties in model predictions. Additionally, extending the classical LV system to include ecological complexities like seasonal effects, environmental changes, and stochastic influences could provide a more realistic representation of predator-prey dynamics. Further comparisons between numerical methods, including higher-order solvers and hybrid AI-numerical techniques, could enhance computational efficiency and stability. Investigating these advanced modeling strategies would offer deeper insights into ecosystem interactions and contribute to better wildlife conservation and resource management.

References

[1] H. A. Adamu, "Mathematical analysis of predator-prey model with two preys and one predator," *International Journal of Engineering and Applied Sciences*, vol. 5, no. 11, pp. 17–23, 2018.

[2] R. M. Anderson and R. M. May, *Infectious diseases of humans: dynamics and control.* Oxford university press, 1991.

[3] P. Antoniou and A. Pitsillides, "Congestion control in autonomous decentralized networks based on the lotka-volterra competition model," in *International Conference on Artificial Neural Networks.* Springer, 2009, pp. 986–996.

[4] ——, "A bio-inspired approach for streaming applications in wireless sensor networks based on the lotka–volterra competition model," *Computer Communications*, vol. 33, no. 17, pp. 2039–2047, 2010.

[5] ——, "Congestion control in wireless sensor networks based on the lotka volterra competition model," in *Biologically Inspired Networking and Sensing: Algorithms and Architectures.* IGI Global, 2012, pp. 158–181.

[6] M. Beals, L. Gross, and S. Harrell, "Predator-prey dynamics: Lotka-volterra," 1999, accessed: 2025-03-09. [Online]. Available: http://www.tiem.utk.edu/~gross/bioed/bealsmodules/predator-prey.html

[7] W. E. Boyce and R. C. DiPrima, *Elementary differential equations and boundary value problems*, 7th ed. John Wiley & sons, 2001.

[8] J. BREARLEY and A. Soudack, "Approximate solutions to the lotka-volterra competition equations," *International Journal of Control*, vol. 27, no. 6, pp. 933–941, 1978.

[9] E. M. Cardona, C. A. Ramírez-Vanegas, and J. R. G. Granada, "Numerical Solution of the Lotka-Volterra Stochastic Differential Equation," *Statistics, Optimization & Information Computing*, 2025. [Online]. Available: http://www.iapress.org/index.php/soic/article/view/2307

[10] J. D. Davis, D. V. Olivença, S. P. Brown, and E. O. Voit, "Methods of quantifying interactions among populations using lotka-volterra models," *Frontiers in Systems Biology*, vol. 2, p. 1021897, 2022.

[11] O. Diekmann and J. A. P. Heesterbeek, *Mathematical epidemiology of infectious diseases: model building, analysis and interpretation.* John Wiley & Sons, 2000, vol. 5.

[12] F. N. Egerton, "History of ecological sciences, part 54: Succession, community, and continuum," *Bulletin of the Ecological Society of America*, vol. 96, no. 3, pp. 426–474, 2015.

[13] C. M. Evans and G. Findley, "Analytic solutions to the lotka-volterra model for sustained chemical oscillations," 1998.

[14] M. E. Gilpin, "Do hares eat lynx?" *The American Naturalist*, vol. 107, no. 957, pp. 727–730, 1973.

[15] D. Hundley, "Math 250: Introduction to mathematical logic," 2003, accessed: 2025-06-07. [Online]. Available: http://people.whitman.edu/~hundledr/courses/M250F03/M250.html

[16] C. Jost, G. Devulder, J. A. Vucetich, R. O. Peterson, and R. Arditi, "The wolves of isle royale display scale-invariant satiation and ratio-dependent predation on moose," *Journal of Animal Ecology*, pp. 809–816, 2005.

[17] M. Kawira, C. G. Ngari, and S. Karanja, "A theoretical model of corruption using modified lotka volterra model: A perspective of interactions between staff and students," 2020.

[18] A. J. Lotka, "Analytical note on certain rhythmic relations in organic systems," *Proceedings of the National Academy of Sciences*, vol. 6, no. 7, pp. 410–415, 1920.

[19] A. Marasco, A. Picucci, and A. Romano, "Market share dynamics using lotka–volterra models," *Technological forecasting and social change*, vol. 105, pp. 49–62, 2016.

[20] R. M. May and R. M. Anderson, "The transmission dynamics of human immunodeficiency virus (hiv)," *Philosophical Transactions of the Royal Society of London. B, Biological Sciences*, vol. 321, no. 1207, pp. 565–607, 1988.

[21] A. Meça, A. Uka, and M. Ali, "Parameter estimation of three cell interaction using lotka volterra differential equations," in *2021 International Conference on Computing, Networking, Telecommunications & Engineering Sciences Applications (CoNTESA).* IEEE, 2021, pp. 12–18.

[22] V. Noonburg, "A neural network modeled by an adaptive lotka-volterra system," *SIAM Journal on Applied Mathematics*, vol. 49, no. 6, pp. 1779–1792, 1989.

[23] M. Passarella, "The paradox of tranquility revisited. a lotka-volterra model of the financial instability," *Rivista italiana degli economisti*, vol. 15, no. 1, pp. 69–104, 2010.

[24] D. Y. Regalado and E. C. Castillano, "Approximate analytic solution to the lotka-volterra predator â€"prey differential equations model," *Journal of Higher Education Research Disciplines*, vol. 4, no. 1, pp. 38–47, 2019.

[25] J. Toyoda and H. Saitoh, "Proposal of an open-electric-energy-network (oeen) to realize cooperative operations of iou and ipp," in *Proceedings of EMPD'98. 1998 International Conference on Energy Management and Power Delivery (Cat. No. 98EX137)*, vol. 1. IEEE, 1998, pp. 218–222.

[26] B. H. Tsai, "Forecasting ic design industrial clustering from taiwan to mainland of china using four-dimensional competitive lotka-volterra equations," *International Journal of Innovative Computing, Information and Control*, vol. 20, no. 1, pp. 249–261, 2024.

[27] V. Volterra, "Fluctuations in the abundance of a species considered mathematically," *Nature*, vol. 118, no. 2972, pp. 558–560, 1926.

[28] S.-Y. Wang, W.-M. Chen, and X.-L. Wu, "Competition analysis on industry populations based on a three-dimensional lotka–volterra model," *Discrete Dynamics in Nature and Society*, vol. 2021, no. 1, p. 9935127, 2021.

[29] P. J. Wangersky, "Lotka-volterra population models," *Annual Review of Ecology and Systematics*, vol. 9, pp. 189–218, 1978.