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Exploring Fibonacci Algebraic Equations and Their Application

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Abstract

This article explores the sequence's generating functions, algebraic equations regulating it, and relationships to abstract algebra. Fibonacci algebra shows how patterns, numbers, and equations interact, from quadratic equations and generating functions to matrix algebra and beyond. Its simplicity and the surprising breadth of its linkages to other areas of mathematics are what give it its ongoing appeal. The study of Fibonacci algebraic equations is not just an exploration of an ancient sequence but a gateway to profound mathematical structures and applications. Whether through its connections to the golden ratio, matrix algebra, or advanced number theory, Fibonacci algebra continues to inspire and challenge mathematicians, offering a timeless bridge between elementary and advanced mathematics.

Keywords: Fibonacci number and sequence, Algebraic quadratic equation, Natural Phenomenon

Introduction

The Fibonacci sequence is one of the most captivating constructs in mathematics, characterized by its elegant recursive structure and deep connections to various domains, including nature, art, and science. Since the earliest days of the ancient Greeks, mathematicians have been committed to the explanation of nature (Grigas,2003 AD).

More than a pastime, finding patterns in nature helps us create mathematical tools that allow us to estimate. Interestingly, mathematical designs are also seen in many biological species. It is an illustration of a typical and intriguing natural succession Bicknell, M. (1973 AD). Nature uses it in a variety of ways. Numerous biological contexts, such as tree branching and flower petal patterns, may contain it. In terms of business and finance, it has to do with whole and decimal numbers

One of the "greatest European mathematicians of the Middle Ages" was Fibonacci. His tomb is situated at the foot of Pisa Cathedral's cemetery tower. He died in 1240 AD, having been born around 1170 AD. Fibonacci grew up among the Moors and received his schooling in North Africa before making a lot of trips along the Mediterranean coast. He then visited traders, helping them with their business and teaching them different mathematical abilities. The "Hindu-Arab" approach was clearly superior than the others (Grigas,2003 AD).

The Hindu-Arab system was initially introduced to Europe by Fibonacci. Ten base numerals are employed, each consisting of a zero sign and decimal points: 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. (Bicknell,1973).

The Fibonacci sequence is one of the most celebrated sequences in mathematics, known for its elegant recursive structure and surprising applications across diverse fields, from nature to computer science. The sequence is defined as:

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$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + f_{n-2} \text{ for } n \geq 2. \quad (1)$$

The well-known Golden Proportion $\alpha = \frac{1+\sqrt{5}}{2}$ create often in nature, is the root of this equation. Many objects alive in the natural world that own pentagonal symmetry like marine stars, inflorescences of many flowers, and phyllotaxis objects have a numerical explanation given by the Fibonacci numbers which are themselves based on the Golden Proportion.

Fibonacci numbers are always zero at the beginning and one at the end. There are many amazing natural occurrences that exhibit the Fibonacci sequence. Fibonacci numbers were discovered in the 13th century by the Italian mathematician Leonardo Fibonacci (Dasdan.2018AD). The origins and uses of Fibonacci numbers were thoroughly explained by (Bortner, & Peterson, 2016, AD). This article investigates into the algebraic equations governing the sequence, its generating functions, and its connections to abstract algebra.

In the last few years, the Golden Proportion has played an increasing role in contemporary physical research by El Naschie. (1992, 1993, 1994, 1998,2002,2004AD), Grigas. A. (2003 AD), Kumar, et. al. (2024AD), (2022AD), Mauldin, et al. (1986AD), Stakhov, (1989AD), (2005AD).

Novelty

The originality of Fibonacci sequences frequently comes from creative applications, discoveries, or interpretations, even when the patterns themselves are old and well-known. Fibonacci sequences have been used in a variety of fields over time, including mathematics, computer science, art, and even business. Recent research may examine new uses of Fibonacci sequences in algorithmic design, data compression, or cryptography. This work will explore Fibonacci Algebraic Equations and their application.

Innovation

Fibonacci sequences may be innovative in a number of ways. In order to efficiently build or analyze huge Fibonacci sequences, it can entail creating new computing techniques or algorithms. Combining Fibonacci sequences with other mathematical ideas or theories to address issues that haven't been addressed before may also lead to innovations. Additionally, a better comprehension of Fibonacci patterns and structures is made possible by advancements in visualization tools.

Significance

This work is important because they are widely used and applicable in many different fields. Fibonacci sequences in mathematics are related to algebra, combinatorics, and number theory. They also show up in the way leaves are arranged on stems and how trees branch. Gaining knowledge of Fibonacci sequences can advance theoretical understanding and practical applications by illuminating the fundamental mathematical ideas behind these occurrences.

Preliminary Definitions

Fibonacci Sequences

Each number in the Fibonacci sequence is equal to the sum of the two sequences that came before it. The first two numbers, 0 and 1, are used initially. In mathematics, this sequence is well-known. Both plants and animals can have Fibonacci numbers. Some refer to these figures as the global rule of nature. Thus, the series' initial numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, and so forth.

0,1,1,2,3,5,8,13,21,34,55,89,144,233,377, are called Fibonacci numbers. The Fibonacci number of is n.

$F_1 = F_2 = 1$ opening condition

$F_n = F_{n-1} + F_{n-2}$, $n \geq 3$, repetition relation and it can also represent as:

$$F_n = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ F_{n-1} + F_{n-2} & n > 1 \end{cases}$$

One important property of Fibonacci numbers is that if

$\alpha = \frac{(1+\sqrt{5})}{2}$, then $\alpha^{n-2} < F_n < \alpha^{n-1}$, $n \geq 3$. The irrational number α is called golden ratio.

Lucas numbers: Lucas numbers are a sequence of numbers, similar to the Fibonacci numbers. The French mathematician François Édouard Anatole Lucas, who investigated these sequences of numbers in the late 19th century, is honored by his name (Singh. & Hemachandra,1986AD). The Lucas number sequence is defined as follows:

$L_n = L_{n-1} =$ with early conditions

$L_0 = 2$, $L_1 = 1$

The Lucus numbers, 1, 3, 4, 7, 11, ⋯, are associated with the Fibonacci numbers [5]

$L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$, where n 3,

Just similar Fibonacci numbers, Lucas numbers have motivating properties and connections to various mathematical ideas. They look in nature, art, and many other areas.

Fibonacci Quadratic Equation

Let $\overline{AB}$ is a line segment and we have to find a point C on it such a way that the length of the greater part is the mean proportional between the length of the whole segment.

$$\frac{AB}{AC} = \frac{AC}{CB},$$

Where AB≠ 0, AC ≠ 0, CB≠ 0. Let AB = x where x > 0. Then it can be written

$$x = \frac{AB}{AC} = \frac{AC+BC}{AC} = \frac{AC}{AC} + \frac{BC}{AC} = 1 + \frac{1}{\frac{AC}{BC}} = 1 + \frac{1}{\frac{AB}{AC}} = 1 + \frac{1}{x}$$

$$\therefore\ x = 1 + \frac{1}{x} \qquad (2)$$

On solving, $x^2 - x - 1 = 0$ is a quadratic equation in x. The two roots of this quadratic are

$$\frac{AB}{AC} = \frac{1+\sqrt{5}}{2} \qquad (3)$$

The numerical value $\alpha = 1.618$ and $\beta = -.618$.

This computation verifies that the construction does definitely C on $\overline{AB}$

$$\frac{AB}{AC} = \frac{1+\sqrt{5}}{2} \qquad (3)$$

Let α is a root of equation (1) then

$$\alpha^2 = \alpha + 1 \qquad (4)$$

Multiplying both sides of equation (4) by α^n (n can be any integer) gives

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n.$$

Again, let we let $u_n = \alpha^n, n \geq 1$, then $u_1 = \alpha$ and $u_2 = \alpha^2$, and we have the sequence

$$\alpha,\ \alpha^2 = a + 1,\ a^3 = a^2 + a, \qquad (5)$$

which satisfies the recursive formula $u_n = u_{n-1} + u_{n-2}$, for n>2. Similarly, the other sequence

$$\beta^{n+2} = \beta^{n+1} + \beta^n, \qquad (6)$$

$$\beta,\ \beta^2 = \beta + 1,\ \beta^3 = \beta^2 + \beta,$$

also satisfies recursive formula $u_n = u_{n-1} + u_{n-2}$, for n>2.

$$\beta = 1 \text{ and } \alpha - \beta = \sqrt{5} \qquad (7)$$

If we now subtract the members of equation (6) from the members of equation (5) and divide each member of the resulting equation by $\alpha - \beta (= \sqrt{5} \neq 0)$, we find

$$\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

Let $u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, n \geq 1$, then we have

$$u_{n+2} = u_{n+1} + u_n$$

and

$$u_1 = \frac{\alpha - \beta}{\alpha - \beta} = 1$$

$$u_2 = \frac{\alpha^2 - \beta^2}{\alpha - \beta} = \frac{(\alpha - \beta)(\alpha + \beta)}{\alpha - \beta} = \frac{(\sqrt{5})(1)}{\sqrt{5}} = 1$$

Sequence u_n is exactly the Fibonacci sequence.

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ n = 1,2,3, \ldots \qquad (8)$$

is the Binet form of the Fibonacci numbers after the French mathematician Jacques-Phillipe-Marie Binet (1786-1856AD).

Since the relationship of the roots, α and β, of the equation (F),

$$x^2 - x - 1 = 0$$

to the Fibonacci numbers, we shall call equation (3) the Fibonacci quadratic equation.

We shall call the positive root of (4) is

$\alpha = \frac{1+\sqrt{5}}{2}$ is the Golden ratio.

Algebraic Representation of the Fibonacci Sequence

Fibonacci numbers can be represented algebraically through generating functions and matrix methods. One of the most popular methods is the use of the closed-form expression, known as Binet's Formula:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ n = 1,2,3, \ldots$$

where is the golden ratio. This formula allows for direct computation of any Fibonacci number without recursion.

Fibonacci Algebraic Equations

Algebraic equations involving Fibonacci numbers often exploit their recursive nature. For example:

Quadratic Relationships: This relationship connects consecutive Fibonacci numbers in a non-linear way and highlights their alternating properties.

Sum of Fibonacci Numbers: The sum of the first Fibonacci numbers can be expressed as:

Even and Odd Properties: Fibonacci numbers satisfy modular arithmetic properties. For instance, every third Fibonacci number is even, and every fourth Fibonacci number is divisible by 3.

Some Fibonacci algebra

The two roots of the equation (2) are

$\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ of the quadratic equation

$x^2 - x - 1 = 0$.

From equation (6) and (7)

$\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. Also, $\alpha + \beta = 1$ and $\alpha - \beta = \sqrt{5}$.

Furthermore,

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n \quad (9)$$

and

$$\beta^{n+2} = \beta^{n+1} + \beta^n \quad (10)$$

and by using equations (9) and (10), the Fibonacci numbers can be expressed in a form that is called Binet form:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}},\ n = 1,2,3, \ldots \quad (11)$$

Now adding equation (10) and equation (9),

$$(\alpha^{n+2} + \beta^{n+2}) = (\alpha^{n+1} + \beta^{n+1}) + (\alpha^n + \beta^n)$$

Now let $u_n = \alpha^n + \beta^n$, then we have

$$u_{n+2} = u_{n+1} + u_n$$

and

$$u_1 = \alpha + \beta = 1$$
$$u_2 = \alpha^2 + \beta^2 = \alpha + 1 + \beta + 1 = (\alpha + \beta) + 2 = 1 + 2 = 3$$

The sequence u_n is the sequence of Lucas numbers and so we must have a Binet form for the Lucas numbers:

$$L_n = \alpha^n + \beta^n,\ n = 1,2,3, \ldots \quad (12)$$

The comparison of the Fibonacci numbers and the Lucas numbers: are described as:

F_1	F_2	F_3	F_4	F_5	F_8	F_7	F_8	F_9	F_{10}	…
1						3	1	4	5	…
1				1	8	9	7	6	23	…
L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	…

It can be written as

$F_1 + F_3 = L_2$, $F_2 + F_4 = L_3$ and so on.

Example 2. Show that $L_n = F_{n-1} + F_{n+1}$, n≥1.

$L_n = F_{n-1} + F_{n+1}$

since $F_{n+1} = F_n + F_{n-1}$

$L_n = F_n + 2F_{n-1}$

It is easily verified, $L_6 = F_6 + 2F_5$, and so on.

Expression of F_n and L_n in terms of α^n and β^n

F_n and L_n can be expressed in terms of α^n and β^n and α^n and β^n also expressed in terms of F_n and L_n vice versa. If $\alpha - \beta = \sqrt{5}$, then from the Binet forms

$\sqrt{5}f_n = \alpha^n - \beta^n$ (13)

$L_n = \alpha^n + \beta^n$

(14)

Adding equations (12) and (13) gives

$2\alpha^n = L_n + \sqrt{5}F_n$

$\alpha^n = \frac{L_n+\sqrt{5}F_n}{2}$

Subtracting equation (14) from (13) gives

$\beta^n = \frac{L_n-\sqrt{5}F_n}{2}$

F_0 can be expressed as $F_2 - F_1$ and L_0 as $L_2 - L_1 = 3 - 1 = 2$.

Since $\alpha^0 + \beta^0 = 1 + 1 = 2$ and $L_1 = \alpha + \beta = 1$ then L_n is expressed as

$L_n = L_1F_n + L_0F_{n-1}$ (15)

In this way F_{-1}, L_{-1} can also define by applying the formulas

$F_{n-1} = F_{n+1} - F_n$

$L_{n-1} = L_{n+1} - L_n$

frequently. Thus, we have:

…	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	…
…	-3	2	-1	1	0	1	1	2	3	…
…	7	-4	3	-1	2	1	3	4	7	…
…	L_{-4}	L_{-3}	L_{-2}	L_{-1}	L_0	L_1	L_2	L_3	L_4	…

We can derive a formula for $F_{-n}, n > 0$, by assuming that the Binet form also holds for negative values of the exponents

$$F_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{\left(\frac{1}{\alpha}\right)^n - \left(\frac{1}{\beta}\right)^n}{\alpha - \beta} \tag{16}$$

Example 3. Verify $\alpha\beta = -1$.

$\frac{1}{\alpha} = -\beta$ and $\frac{1}{\beta} = -\alpha$

So,

$$F_{-n} = \frac{(-\beta)^n - (-\alpha)^n}{\alpha - \beta} = \frac{(-1)^n(\beta^n - \alpha^n)}{\alpha - \beta} = (-1)^{n+1}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)$$

and

$$F_{-n} = (-1)^{n+1} F_n \tag{17}$$

Generating Functions

Generating functions encode sequences into algebraic expressions, making them a powerful tool in studying Fibonacci numbers. The generating function G(x) for the Fibonacci sequence is:

$$G(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$

This function reveals how the Fibonacci sequence connects to rational functions and provides insights into the sequence's asymptotic growth, zeros, and poles in the complex plane.

Applications of the Generating Function

Series Expansion: The Taylor series of G(x) recovers the Fibonacci sequence coefficients.

Algebraic Manipulations: Enables derivation of relations like summing specific Fibonacci terms or exploring moduli.

Applications and Implications

Applications in Algebra

Polynomials: Fibonacci numbers often appear in polynomial identities. For instance: is the minimal polynomial of the golden ratio, which is deeply tied to Fibonacci sequences. Extending this idea, Fibonacci polynomials are defined recursively and provide a generalization:

Matrix Algebra: The Fibonacci sequence can be represented through matrix exponentiation. The fundamental relation: allows for efficient computation of Fibonacci numbers using linear algebra techniques.

Diophantine Equations: Fibonacci numbers are solutions to certain types of Diophantine equations. For instance: has solutions derived from Fibonacci numbers when and are integers.

Advanced Topics

Fibonacci in Abstract Algebra: In group theory, the additive group of integers modulo exhibits periodic behavior of Fibonacci numbers. These properties can be explored to study cyclic groups and their generators.

Connection with Lucas Numbers: Lucas numbers, defined as are closely related to Fibonacci numbers and satisfy similar algebraic equations. Exploring their relationships opens new pathways in number theory.

Generalizations: The Fibonacci sequence can be extended to higher dimensions or generalized using different recurrence relations, such as the Tribonacci sequence: Such generalizations have applications in coding theory and combinatorics.

Cryptography: Fibonacci sequences are used in pseudorandom number generators and secure key exchanges.

Algorithm Design: Algorithms based on Fibonacci heaps leverage their properties for efficient data management.

Geometry and Art: The connection between Fibonacci numbers and the golden ratio influences proportions in design and architecture.

Advancement

Improvements in processing capacity, mathematical theory, and multidisciplinary cooperation frequently accompany developments in Fibonacci sequences. Researchers may now investigate bigger Fibonacci sequences and conduct more thorough analyses of their features because to the development of high-performance computers. Fibonacci sequences and their extensions may be studied with new tools thanks to developments in mathematical concepts like number theory and algebraic geometry. By bringing together specialists from several disciplines to address challenging Fibonacci sequence challenges, interdisciplinary collaboration fosters creativity.

Conclusion

The Fibonacci sequence transcends its simple definition, offering a gateway into profound algebraic structures and applications. From quadratic equations and generating functions to matrix algebra and beyond, Fibonacci algebra demonstrates the interplay between patterns, numbers, and equations. Its enduring appeal lies in its simplicity and the unexpected depth of its connections to various mathematical domains.

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