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Applications of Fourier Series and Fourier Transformation

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Abstract

This paper investigates into the application of fourier transformation and series, which converts time domain signal to frequency domain signals, at which signals can be analyzed. Unlike Laplace transform, Fourier Transforms does not have full S plane, it just have the frequency j2πf plane. Fourier Transforms helps to analyze spectrum of the signals, helps in find the response of the LTI systems. (Continuous Time Fourier Transforms is for Analog signals and Discrete time Fourier Transforms is for discrete signals). Discrete Fourier Transforms are helpful in Digital signal processing for making convolution and many other signal manipulations. Overall, the paper will conclude the impact of Fourier Transforms in life.

Key Words: [Fourier, Transforms, coefficient, convergence, integral]

1. Introduction:

1.1. Fourier Series: Fourier series decomposes a periodic function into a sum of sines and cosines with different frequencies and amplitudes, was introduced by Joseph Fourier. The notion of Fourier series in the work of Euler and D. Bernoulli on vibrating strings, but the theory of Fourier series truly began with the profound work of Fourier on heat conduction. Fourier deals with the problem of describing the evolution of the temperature T(x, t) of a thin wire of length π, stretched between x=0 and x= π, with a constant zero temperature at the ends: T(0, t)=0 and T(π, t)=0 and at initial temperature expanded in a series of sine function as below:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{.............................(1)}$$

Where b_n is the coefficient and value can be calculated as

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \qquad \text{..........................(2)}$$

The Fourier sine series, defined in Equation (1) is a special case of a more general concept.

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If the function f has period 2π , then its Fourier series is express as

$$c_0 + \sum_{i=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ ....(3)}$$

Where $\sum_{i=1}^{\infty} a_n \cos nx = a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots\ldots\ldots\ldots$ and

$$\sum_{i=1}^{\infty} b_n \sin nx = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots\ldots\ldots\ldots$$

With Fourier coefficient and the value of coefficient is defined by integral

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \text{ ....(4)}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \text{ ....(5)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \text{ ....(6)}$$

According to Euler, we can use complex exponential $e^{i\theta}$ satisfied ,

Therefore, we can express cosθ and sinθ as follows:

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \text{ ....(7)}$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \text{ ....(8)}$$

Hence from 4, 5, 6, 7 & 8 we write (3) as

$$c_0 + \sum_{i=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx}) \text{ ....(9)}$$

With c_n defined for all integers n by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \quad \ldots\ldots\ldots\ldots\ldots\ldots..(10)$$

The series in (9) is usually written in the form

$$\sum_{i=\infty}^{\infty} c_n e^{inx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(11)$$

If series in (11) is converges to f then

$$f(x) = \sum_{i=\infty}^{\infty} c_n e^{inx} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(12)$$

Then f is being expressed as a superposition of elementary functions having frequency n/2π with coefficient or amplitude c_n.

There are many ways to interpret the meaning of Eq. (12). Investigations into the types of functions allowed on the left side of (12), and the kinds of convergence considered for its right side, have fueled mathematical investigations by such luminaries as Dirichlet, Riemann, Weierstrass, Lipschitz, Lebesgue, Fejer´, Gelfand, and Schwartz.

Fourier series are of great importance in both theoretical and applied mathematics, for orthonormal families of complex valued functions $\{\varphi_n\}$, Fourier Series are sums of the φ_n that can approximate periodic, complex valued functions with arbitrary precision. Fourier series are especially attractive because uniform convergence of the Fourier series is guaranteed for continuous, bounded functions and many more. Periodic Bernoulli functions play an important role in several mathematical results such as the general Euler-McLaurin summation formula. A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions[1-5]. A function f(x) is even if f(x) = f(−x) and odd if f(x) = −f(−x) for all x. Most useful of these rules are the following: Product of two even functions is even, two odd functions is even, an even and odd function is odd, derivative of an even function is odd and an odd function is even.

2. Pseudo-Fourier Transform: The pseudo-Fourier cosine transform based on the semiring

([0,1], ϕ, Θ) of a measurable function is $f : R \to [0,1]$

$$F_c^{\phi}[f(x)](\omega) = g^{-1}\left(\frac{1}{2^{1/2}}\right)\Theta \int_{[-\pi,\pi]}^{\phi} g^{-1}(\cos(\omega x)\ \Theta E_p(x)dx \quad \ldots\ldots\ldots13$$

for every real number ω (if the right side exists). The pseudo-Fourier sine transform based on the semiring ([0,1], ϕ, Θ) of a measurable function is

$$F_s^{\phi}[f(x)](\omega) = g^{-1}\left(\frac{1}{2^{1/2}}\right)\Theta \int_{[-\pi,\pi]}^{\phi} g^{-1}(\sin(\omega x)\ \Theta O_p(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots...14$$

for every real number ω.

3. Inverse Pseudo-Fourier Transform: If we are able to convert from pseudo-Fourier images back to functions, i.e. if there exists the inverse transformation. The simple calculation leads to

$$f(\omega) = f(gof)(\omega) = \left(\frac{1}{2^{1/2}}\right)\int_{-\infty}^{\infty} gof(t)e^{-i\omega t}dt \quad \ldots\ldots\ldots\ldots\ldots\ldots.15$$

Where f is the classical Fourier transform [6].

5. Parseval's Identity: If a function f(x) converges uniformly in (c, c + 2l), then a special identity known as the Parseval's identity is applicable and represented as,

$$\frac{1}{2l}\int_{c}^{c+2l}[f(x)]^2 dx = \frac{a_0^2}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \ldots\ldots\ldots..16$$

Here (16) is known as the Parseval's Identity. The generalized form of the Fourier Series in the interval (- l, l) is expressed as,

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots...17$$

Now, if the function f(x) is even, then the coefficient b_n will turn out to be zero within the limits (- 1 , 1) then (17) can be represented as,

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right)\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots..18$$

Thus, (18) now does not have any sine term in the expression and this is called the *Half Range Cosine Series.* Similarly, if the function f(x) is an odd function, then the coefficients

a_0 and a_n turn out to be zero within the limits (- l, l) and the series would then be represented as,

$$f(x) = \sum_{i=1}^{\infty}\left(b_n \sin\left(\frac{n\pi x}{L}\right)\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots..19$$

Thus, (20) the series now does not have any cosine term in the expression and this is called the *Half Range of Sine Series*.

6. Fourier Transformer: Fourier transform defines a relationship between a signal in the time domain and its representation in the frequency domain. The Fourier transform decomposes a function into oscillatory functions. Fourier transform is applied in solving differential equations since the Fourier transform is closely related to Laplace transformation. Fourier transform is also used in nuclear magnetic resonance (NMR) and in other kinds of spectroscopy. If we define Fourier series of a periodic function f(x) in the interval $-a/2 \leq x < a/2$, over the entire real line, $x \in R$, by taking the limit $a \to \infty$ i.e.

$$f(x) = \lim_{x\to\infty}\sum_{n=\infty}^{\infty} e^{k_n x} f_n \text{ where } k_n = n\Delta k,\ \Delta k = 2\pi n/\ a \ldots\ldots\ldots20$$

As $a \to \infty$, the wave-number quantum Δk goes to zero. Hence, the set of values of k_n turns into a continuum, and we can replace the discrete sum with an integral over the values of k_n. To do this, we multiply the summation by a factor of $(\Delta k/2\pi)/(\Delta k/2\pi) = 1$

$$f(x) = \lim_{x\to\infty}\sum_{n=\infty}^{\infty}\frac{\Delta k}{2\pi}e^{k_n x}\frac{2\pi f_n}{\Delta k} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..21$$

If we now define $F(k_n) = \dfrac{2\pi f_n}{\Delta k}$, ………………...22

Then the sum becomes

$$f(x) = \lim_{x\to\infty}\sum_{n=\infty}^{\infty}\frac{\Delta k}{2\pi}e^{k_n x}F(k_n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.23$$

This limiting expression matches the basic definition of an integral become

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik_n x} F(k) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 24$$

The factor of 2π is essentially arbitrary, Our choice corresponds to the standard definition of the Fourier transform.

$$F(k_n) = \lim_{x\to\infty} \frac{2\pi f_n}{\Delta k} \ldots\ldots\ldots\ldots\ldots 25$$

$$F(k_n) = \lim_{x\to\infty} \frac{2\pi}{2\pi / a} \frac{1}{a} \int_{-a/2}^{a/2} dx e^{-k_x x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots 26$$

$$F(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 27$$

Hence, we have a pair of equations called the Fourier relations

$$F(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 28$$

$$f(k) = \int_{-\infty}^{\infty} dx e^{ikx} F(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 29$$

The equation (28) is the Fourier transform, and the equation (29) is called the inverse Fourier transform. These relations state that if we have a function f(x) defined over x ∈ R, then there is a unique counterpart function F(k) defined over k ∈ R, and vice versa. The Fourier transform converts f(x) to F(k), and the inverse Fourier transform does the reverse.

6.1. Basic properties of the Fourier transform

The Fourier transform is linear

Performing a coordinate translation on a function causes its Fourier transform to be multiplied by a "phase factor":

The Fourier transform of a real function must satisfy the symmetry relation $F(k) = F^*(-k)$, meaning that the Fourier spectrum is symmetric about the origin in k-space.

When you take the derivative of a function, that is equivalent to multiplying its Fourier transform by a factor of ik

Scaling, Time/Space Shift, Frequency Shift, Modulation, Differentiation, Integration, Transform of a transform, Central ordinate, Equivalent width, Convolution, Derivative of a convolution, Cross-correlation, Auto-correlation, Parseval/Rayleigh [7-8].

Discrete Fourier Transform (DFT) is more convenient to handle for the frequency analysis of discrete time signals. A finite duration sequence of length L has a Fourier Transform in interval $0 \leq \omega \leq 2$ is given by,

$$F(s) = \sum_{n=0}^{L-1} f(n) e^{i\omega t} \quad \ldots\ldots\ldots\ldots..30$$

Where the upper and lower indices in the summation reflect the fact that f(n) = 0 outside the range $0 \leq n \leq L-1$. When we sample F(s) at equally spaced frequencies $2\pi k/$, k=0, 1… N-1, where $N \geq L$,

$$F(k) = \sum_{n=0}^{L-1} f(n) e^{-\frac{2\pi i n}{N}} \quad \ldots\ldots\ldots\ldots...31$$

Also from (10) we can convert a finite sequence into a set of frequency samples of length N and the frequency samples are obtained by evaluating the Fourier Transform into a set of equally spaced frequencies, equation (10) is thus called the Discrete Fourier Transform of f(n), the sequence f(n) from the frequency samples is given by,

$$f(n) = \frac{1}{N} \sum_{n=0}^{L-1} F(s) e^{\frac{2\pi i n}{N}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots...32$$

and (32) is called the Inverse Discrete Fourier Transform (IDFT) [9].

7.1. History and Report of FS and FT

In 1747 Trigonometric series of a function derived from movement of planets, used method derived in 1729 Method for interpolation. In 1753 The first series decomposition of a signal is due to Daniel Bernoulli who showed that Daniel Bernoulli. In 1754 Series in cosine functions of the reciprocal value of the mutual distance of two planets Jean Le Rond d'Alambert. Alexis Claude Clairaut 1757 Cosine series of a function derived in a study of the perturbations caused by the Sun. Riesz-Fischer theorem in 1907 [10].

In the Fourier Transform (FT) area, the advancements of oversampling, computerized sifting, and clamor molding are generally received for smothering the quantization commotion. Some popular techniques used for PQ analysis are: STFT, GT, HHT, KFs, PMs, WT, ST, FRFT, DFRFT and many more [11].

Fourier developed the theory of the series, and applied it to the solution of boundaryvalue problems in partial differential equations, Biology and Medicine, Photograph, Rapid City, SD, Electrocardiogram, X-Ray Computerized Tomography, Fourier Transform Maps one function to another, An integral transform, DFT, Maps discrete vector to another discrete vector, FFT, 2D FFT and Image Processing, Spatial Frequency in Images, 2D Discrete Fourier Transform, 2D FFT, Convolution and Convolution and so on [12].

The connection between mathematics and music goes back at least as far as the sixth century B.C. with a Greek philosopher named Pythagoras. In 18th century, calculus became a tool, and was used in discussions on vibrating strings. Brook Taylor, who discovered the Taylor Series, found a differential equation representing the vibrations of a string based on initial conditions. Daniel Bernoulli and Leonhard Euler, Swiss mathematicians, and Jean-Baptiste D'Alembert, a French mathematician, physicist, philosopher, and music theorist, were all prominent in the ensuing mathematical music debate. D'Alembert was also led to a differential equation from Taylor's problem of the vibrating string,

$$\frac{\partial^2 y}{\partial x^2} = \alpha \frac{\partial^2 y}{\partial t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 34$$

where the origin of the coordinates is at the end of the string, the x-axis is the direction of the string, y is the displacement at time t. The Fourier series is a solution to the wave equation, and can therefore be used to model sound [13]. Fast Fourier transform algorithm had been published many years ago by Runge and Konig and by Stumpff and described primarily how one could use symmetries of the sine-cosine functions to reduce the amount of computation by factors of 4, 8, or even more. To demonstrate a simple application of the FFT program, data from a strain seismograph of the Rat Island, Alaska, earthquake were Fourier-analyzed by Dr. L. Alsop of IBM in 1966. A conventional program took 1567.8 seconds to compute the period gram or estimated power spectrum which is shown in Fig. 9. The same task took 2.4 seconds with the FFT program and gave more accurate results [14]. The degree of approximation of 2π-periodic functions of two variables, defined on $T^2 = [-\pi,\pi] \times [-\pi,\pi]$ and belonging to certain Lipschitz classes, by means of almost Euler summability of their Fourier series. The concept of almost convergence of sequences was introduced and studied by G.G. Lorentz in 1948. Móricz and Rhoades extended the definition of almost convergence to double sequences of real numbers $\{x_{mn}\}$, almost converging to L. Móricz and Xianlianc Shi studied the rate of uniform approximation of a 2π-periodic continuous function. Khan and Ram determined the degree of approximation for the functions belonging to the class Lip [15].

8. The Classical Fourier Transform for Continuous Time Signals

Properties of the Continuous Time Fourier Transform

- Linearity (superposition).

- Time shifting.
- Frequency shifting.
- Time domain convolution.
- Frequency domain convolution.
- Time differentiation.
- Time integration.
- Fourier Spectrum of the Continuous
- Fourier Transform of Periodic
- Generalized Complex Fourier Transform.

9. The Discrete Time Fourier Transform

a. Properties of the Discrete Time Fourier Transform

- Linearity (superposition).
- Index shifting.
- Frequency shifting.
- Time domain convolution.
- Frequency domain convolution.
- Frequency differentiation.

b. Relationship between the Continuous and Discrete Time Spectra

10. The Discrete Fourier Transform

- Properties of the Discrete Fourier Series
- Fourier Block Processing in Real-Time Filtering Applications
- Fast Fourier Transform Algorithms [16].

Application of FT and FS:

Some of FS Application are given below:

- Mobile Communication Systems.
- Telecommunication.
- Signal Processing
- Approximation Theory
- Control Theory
- Partial Differential equation
- Image processing
- Heat distribution mapping
- Wave simplification
- Light Simplication: Interference, Deffraction etc.

- Radiation measurements etc.
- Astronomy
- Geology
- Optics
- Telecommunication
- Voice recognition systems
- Automotive
- Hearing devices
- Medicine – imaging and MRI/CT
- Military
- Graphics and Vision.
- Microbiology.
- Images of repetitive structures.
- Identifying regular patterns.
- Application as one of thousand application help mathematician,control systems,electrical engineers, mechanical engineers, physics ...etc) in our life's.

Applications of Fourier Transform: Some of the Application of FT are given below:

- Designing and using antennas
- Image Processing and filters
- Transformation, representation, and encoding
- Smoothing and sharpening
- Restoration, blur removal, and Wiener filter
- Data Processing and Analysis
- Seismic arrays and streamers
- Multibeam echo sounder and side scan
- VLBI & GPS.
- Synthetic Aperture Radar and Interferometric SAR
- High-pass, low-pass, and band-pass filters
- Cross correlation
- X-ray diffraction
- Light
- Differential Equations and PDEs.
- Radiation from Surface Currents.
- Sound, heat, light, stock prices etc.
- Original 3D molecular structure.

- Location of earthquakes & disaster strike.
- Signal and noise estimation
- Application of the trigonometric polynomial in the integration of ODE
- Structure of the instrument
- Characteristics of sound
- Fourier series & Calculus of variations
- Nanostructured materials
- Trigonometric polynomials and Fourier coefficients determination

Some of the differences between Fourier Series and Fourier Transform:

- Fourier series is an expansion of periodic signal as a linear combination of sines and cosines while Fourier transform is the process or function used to convert signals from time domain in to frequency domain.
- The Fourier series is used to represent a periodic function by a discrete sum of complex exponential, while the Fourier transform is then used to represent a general, non-periodic function by a continuous superposition or integral of complex exponential.
- The Fourier transform can be viewed as the limit of the Fourier series of a function with the period approaches to infinity, so the limits of integration change from one period to $(-\infty,\infty)$.
- Fourier series coefficients of a periodic function are sampled values of the Fourier transform of one period of the function.
- Fourier series decomposes a periodic signal into a sum of an infinite number of harmonics (sine and cosine functions) of different frequencies and amplitudes.
- Fourier transform decomposes a non-periodic signal into an infinite number of harmonics having different frequencies and amplitudes. These frequencies are continuous, no frequency is missing.
- Fourier series are known to exist in sinus-cosinus form, sinus form, cosinus form, complex form.

Some of the Limitations of FT & FS.

- It can be used only for periodic inputs and thus not applicable for aperiodic one.
- It cannot be used for unstable or even marginally stable systems.
- Gibbs Phenomenons leads to a secondary issue that Fourier series are not "efficient" at resolving discontinuous or multi-scale functions.
- Location information is stored in phases and difficult to extract.
- The Fourier transform is very sensitive to changes in the function.
- For signals whose frequencies change in time, Fourier analysis has disadvantages which can be overcome by using a windowing process called the Short Term Fourier Transform.

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