Chaturbhujeshwar Academic Journal (CAJ)	Article Info:
ISSN: 3021-9442 (Print)	Received: Aug. 30, 2024
eJournal Site: https://www.cjmcsarlahi.edu.np/publication	Revised: Sep. 25, 2024
Peer-Reviewed, Open Access Journal	Accepted: Nov.12 , 2024

Comparison and Equivalency of Riemann-integrable with Lebesgue- integrable

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Abstract

This article compares Riemann integrability to Lebesgue integrability. A function must also be Lebesgue-integrable if it is Riemann-integrable, but opposite is not always true. The Lebesgue integral is a more generic and flexible idea that may be applied to a larger range of functions. Lebesgue integration provides a more general framework that expands integration options, particularly for more complicated functions. The equivalence of the two integrals is a difficult concept to grasp, and it is regularly covered in advanced real analysis courses.

Keywords: Riemann integration, Lebesgue integration

Introduction

Riemann integral and Lebesgue integral are two different approaches to defining the concept of integration in calculus. Riemann integral is based on partitioning the domain of integration and approximating integral using sums of function values on these partitions. On the other hand, Lebesgue integral in terms of the measure theory, which provides a more general framework for integration (William,2009).

Riemann-integrable function on a closed and bounded interval must also be Lebesgue-integrable. Riemann-integrable is Lebesgue-integrable, though the inverse is not necessarily true.

Some functions are Lebesgue-integrable but not Riemann-integrable.

Riemann integral uses the concept of limits of Riemann sums and requires the function to be "well-behaved" in a certain sense. The cause for this is due to the many definitions and criteria for integrability. Lebesgue integral, on the other hand, is more flexible and can handle a wider class of functions, including those with more complicated behavior (William, 2009).

Riemann integral, which we studied in calculus, is named after German mathematician Bernhard Riemann and is used in a variety of scientific domains, including physics and geometry. Other types of integrals have been created and researched since Riemann's time; nevertheless, they are all extensions of Riemann integral, and comprehending them or appreciating the reasons for their development requires a good comprehension of Riemann integral (William, 2009).

Dang and He (2024) compared the similarities and differences between these two-integration methods in easy way. Their paper gives systematic analysis of basic definitions, concepts, properties, and characteristics. Also, their paper explains the strength and limitations of both integral methods (Dang, & He, 2024).

For example, a function that is continuous almost everywhere (meaning it may have a few points of discontinuity but is continuous elsewhere) is Lebesgue-integrable and it may not be Riemann-integrable if the points of discontinuity are significant enough to affect the Riemann sum [2].

Preliminary Definition

Partition

Partition (divider) P of an interval [a, b] is a set of points $\{x_i : 0 \leq i \leq \eta\}$:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b .$$

Upper and lower sum

Calculus presents upper and lower sums, mainly in relation to Riemann sums. Riemann integrable has upper and lower sums style the same value as the partition's size decreases irrationally.

Let f be a bounded real-valued function on [a, b].

$$a = x_0 < x_1 < \ldots < x_n = b$$

a unit of [a, b]. For every element, upper sum of f

$$S = \sum_{i-1}^{n}(x_i - x_{i-1})m_i$$

and lower sum of f over this element become

$$s = \sum_{i=1}^{n} (x_i - x_{i-1})\, m_i,$$

$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x) , \qquad m_i = \sup_{x_{i-1} \le x \le x_i} f(x)$$

Riemann integral

When a number L has the condition that any E>0, then map f: [a, b] $\rightarrow \mathbb{R}$ for $\mathcal{E} > 0$, $\delta > 0$, defined as:

$$|\sigma - L| < \epsilon. \qquad (2)$$

let σ be Riemann sum of f over || P || < δ so, L Riemann integral:

$$\int_a^b f(x)\, dx = L \qquad (3)$$

Riemann upper and lower integral

Assumed that every upper sum is bounded below by any lower sum and every lower sum is bounded above by any upper sum, real and finite numbers.

Upper Riemann integral is

$$(R) \int_a^{\bar{b}} f(x)dx = \inf S$$

and lower Riemann integral is

$$(R) \int_{\underline{a}}^{b} f(x)dx. = \sup s \qquad (4)$$

$(\mathrm{R})\int_{\bar{a}}^{b} f(x)\,\mathrm{dx}$ (5)

$\psi(x) = c_i$, defined by $x_{i-1} < x < x_i$.

For set of the constants c_i , the step function may integrable,

i.e. $(\mathrm{R}) \int_a^b \psi(x)dx = \sum_{i=1}^{n} c_i\,(x_i - x_{i-1})$ and $(\mathrm{R}) \int_a^{\bar{b}} \psi \ dx = (R) \int_{\underline{a}}^{b} \psi(x)dx.$ (6)

Keep this notion, the following example explain best.

Example: The upper and lower Riemann integrals can be calculated from

f(x) = 0 for rational x, and 1 for irrational x. (7)

Figure 1 [2].

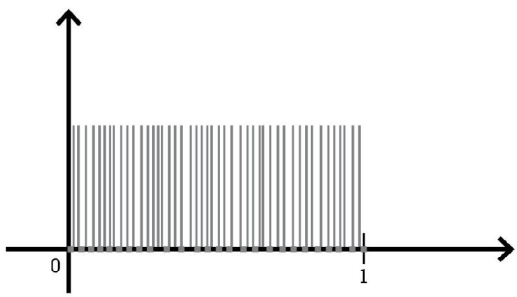

Figure 1: f(x) in Example 1

If we split this function. Since both are dense in real, each subinterval will include both rational and irrational numbers. As a result, each subinterval has a supremum of one and an infimum of zero. According to the definition on the preceding page,

$(\mathrm{R}) \int_a^{\bar{b}} f(x)dx = \text{b-a}, \quad (R) \int_{\underline{a}}^{b} f(x)dx = 0.$ (8)

Upper and lower Riemann integrals are not equal, so this function is not Riemann integrable. This explains one of the Riemann integral's limits. Some discontinuous functions cannot be integrated and areas under them is measured using the Riemann integral [2].

Problems with the Riemann Integration

We discoursed one of the Riemann integral's limits in the preceding scenario. In order to deliver these descriptions, we would find extra kind of integral that is non-Riemann integral functions-containing and not only Riemann integral-like.

By excruciating the domain of an allocated function, we may estimate it using piece-wise constant functions in each sub-interval. Equally, the range of the function is shared by the Lebesgue integral.

A countless comparison with Lebesgue integration is provided by [3]. Let's say we wish to control the total value of coins using both student L (Lebesgue's method) and student R (Riemann's technique). Student R sums the face value of each coin as he selects it at random to find the total. Equally, Student L will count the number of coins and arrange them conferring on their face value. By increasing the sum by its matching face value and combining the results, Student L will calculate the total value [2].

Lebesgue Integration

We want a function of x which is in the measurable set A_i . That function is defined to as the characteristic function. The following statement provides a prescribed definition of the characteristic function and introduction to the simple function.

Definition 1.5 Let set A defined by

$$\chi_A(x) = \begin{cases} 0, & x \in A \\ 1, & \text{otherwise} \end{cases} \qquad (9)$$

is called characteristics function of set A and its linearity is defined as

$\varphi(x) = \sum_{i=1}^{n} a_i \chi_A(x)$ is simple function for measurability of A_i.

A function f, with different values ($a_1, \cdots\cdots, a_n$) then that function is called simple function [7].

$$f(x) = \sum_{i=1}^{n} a_i \chi_A(x) \quad (10)$$

A simple expression

$\varphi(x) = \sum_{i=1}^{n} a_i \chi_A(x)$, where $A_i = \{x \in A: \varphi(x) = a_i\}$. Here A_i are disjoint and a_i are distinct.

Lebesgue integral

The Lebesgue integral is a method of integrating functions with a larger set of characteristics than the Riemann integral. It is useful for functions that are not always continuous or well-behaved, and allows for the integration of functions with complex behaviors, such as infinite values across certain sets or discontinuities. The Lebesgue integral focuses on the function's characteristics over the entire domain, dividing it into measurable sets and assigning values based on the integral's behavior.

Let $\varphi(x) = \sum_{i=1}^{n} a_i x_A(x)$ and $\mu(A_i)$ is finite, the Lebesgue integral is

defined as

$$\int_E \varphi(x) \, dx = \sum_{i=1}^{n} a_i \varphi(A_i).$$

(11)

Upper and lower Lebesgue integrals

Lower Lebesgue integral is the highest lower bound for simple functions, while the upper Lebesgue integral is lowest upper constraint. Understanding these differences is crucial for analyzing Lebesgue integral.

Let f be a finite and limited measurable set defines:

$$I^* f_L = \int_E \quad \inf\{ \varphi(x)dx : \varphi \text{ is simple and}, \varphi \geq f \}, \quad (12)$$

$$I_*(f)_L = \int_E \sup\{\varphi(x)dx : \varphi \text{ is simple and}, \varphi \leq f\}, \tag{13}$$

When both Equation (12) and Equation (13) are equal and f is called Lebesgue integrable,

$$\int_E f(x)\, ds. \tag{14}$$

Equivalency of Riemann and Lebesgue Integration

Riemann integration and Lebesgue integration are two distinct methods for defining and investigating integration in calculus and real analysis. While they both calculate a function's integral, their definitions, characteristics, and circumstances of application differ. Many Riemann integrable functions have corresponding Lebesgue integrals.

Riemann integral equals the Lebesgue integral, and vice versa. However, certain functions are Lebesgue integrable which are not Riemann integrable. The Lebesgue integral is more adaptable and can handle a wider range of functions, making it a more useful tool in some areas of analysis.

Riemann integral divides a function's domain whereas the Lebesgue integral divides its range. Riemann integral's step function has a constant value in each partition's sub-interval, but the Lebesgue integral's simple function produces finitely many measurable sets for each functional value [4]. The Lebesgue integral recovers on the Riemann integral by providing additional generality. In reverse, Riemann integral might suggest Lebesgue integral [5].

Theorem 1. For given a finite f, it is Lebesgue integrable for being Riemann integrable.

$$(R) \int_a^b f(x) \quad dx = \int_{[a,b]} f(x) \quad dx. \tag{15}$$

Proof:

Now, from definition of Riemann upper and lower integral:

$$(R) \int_a^{\bar{b}} f(x) \quad dx = \inf S \qquad \text{and} \qquad (R) \int_{\underline{a}}^{b} f(x) \quad dx = \sup s.$$

All step function is simple with higher sum tops f, that is why, every lower sum $<$ f.

$$\text{(R)} \int_{\underline{a}}^{b} f(x)\text{dx} \leq \text{psu}_{\varphi\leq f} \int_a^b \varphi(x)dx \leq inf_{\psi\geq f} \int_a^b \psi(x)dx \leq (R) \int_a^{\overline{b}} f(x)dx. \quad (16)$$

From Definition of upper and lower Lebesgue integrable, f is Lebesgue integrable.

Theorem 2 Assume that f is a finite measurable set. If f is Lebesgue integrable, then it is measurable. [6, 7].

Proof.

This theorem is not valid or legal without the limitation of a bounded function. A measurable function is not continuously integrable.

Conclusions

While Riemann and Lebesgue integrations are compatible for a wide range of functions, Lebesgue integration provides a more generic framework that broadens the integration possibilities, particularly for more complex functions. The equivalence of the two integrals is not an easy finding, and it is frequently studied in advanced real analysis classes.

Acknowledgments

I would like to sincerely thank the RMC and the entire Chaturbhujeshwar campus community for providing me with the opportunity to publish my manuscript in your peer-reviewed journal. I am also deeply grateful to all researchers whose contributions enriched the literature review and to all participants who provided valuable data for this research.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

Funding

No funding was received for this study or publication. This research was conducted entirely at my own expense.

References

Dang, Q. & He, Y. (2024). Comparative Study of Riemann integral and Lebesgue integral in Calculus. *Theoretical and Nature Sciences*, 31(1),1-8. Doi:10.54254/2753-8318/31.20240731.

Gao, T. T, & Zhang, M. H. (2020). Analysis of the Characteristics and Significance of R (Riemann) Integral. *Journal of Jiamusi University* (Natural Science Edition), 40, 155-158.

Kumar, N. K. (2023). Convergence of Infinite Product By Schlomilch's Method. Tribhuvan University Journal, 38(1),108-118. DOI: https://doi.org./10.3126\tuj.v38i01.56209.

Kunwar, B. L. (2017). Riemann Integral: Significance and Limitations. *Academic Voices*: A Multidisciplinary Journal, 31-34.

Pang, X. C., et al. (2019). Mathematical Analysis. *Higher Education Press*, 186-187.

Russell A. G. (1994). *The integrals of Lebesgue*. Denjoy, Perron, and Henstock, 4. Ameri. Math. Soc.

Zhang, Y. L, et al. (2020). The relationship of Lebesgue Integral and Riemann Integral. *Journal of Jiaozuo Teachers College,* 70-76.

William, F.T, (2016). *Introduction to real analysis*. USA: Pearson Education, 1-563.

Russell A. G. (1994). *The integrals of Lebesgue*. Denjoy, Perron, and Henstock, 4. Ameri. Math. Soc.