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A Solution of a Dispersion Problem for the Concentration of Pollutants in River Water

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Abstract
In this study, one-dimensional advection dispersion equation is solved for the concentration of water pollutant. To solve the differential equation, the Laplace's transform technique has been used assuming no added pollutants on the way of water along the river. From the study, it is found that for various time period and speed, the concentration of the water pollutants decreases continuously, as the distance of the river increases from the point source .

Keywords Dispersion equation, Laplace's transforms, water pollutants, concentration.

1. Introduction

Pollution of water is an enormous problem for the degradation of environment and also a major cause of human health. Water pollution affects the whole individual species, populations and the inborn living group [1].

All living creatures have been facing unhealthy result of polluted water [2]. Living waste substances which have various living and biotransformation steps using solvate oxygen constitutes water pollutants [3]. To predict the concentrations of water pollution, there have been used various analytical and mathematical models.

Using Laplace's transformation, Kumar et al. [4] have solved a one-dimensional diffusion equation for longitudinal semi-infinite homogeneous porous media. In the case of one dimensional transfer in penetrable channel for scale dependent, Huang et al. [5] have given a solution analytically. Using one dimensional diffusion equation, there are various problems and their solutions by different researchers in various time period explaining the dispersion of pollutants in river water. Bhandari [6] has solved a dispersion equation for the concentrations behavior of pollutants with initial rate of pollutants along the river as zero. Paudel et al. [7] have observed the concentration of pollutants and dissolved oxygen in river water. Paudel et al. [8] have studied the variation of concentration of pollutants with various time period.

Some of works are contributed by renowned researchers. Examples includes [Kumar, A. et al. [9], Jaiswal [10], Jaiswal.et al. [11], Mourad [12], Rounds [13], Pimpunchat, et al. [14], Savoric [15], Marine [16], Smith [17], Sirin [18], Wagmare and Kiwne [19], Demuth [20], Yadav et al. [21].

In this research paper, we have given a solution to a one-dimensional differential equation for the diffusion behavior of pollutants in river water using Laplace's transformation method and observed the concentration of pollutants profile in water for different initial pollutants rate.

2. Mathematical formulation and solution

A diffusion equation which is employed to explain the pollutants concentration in flow water for one dimension is given by a partial differential equation [12]:

$$\frac{\partial(AC)}{\partial t} = D_{cx}\frac{\partial^2(AC)}{\partial t^2} - \frac{\partial(UAC)}{\partial x} - k_1\frac{X_o}{X_o+k}AC + Q; \quad 0 \le x < l, \;\; t > 0 \qquad (1)$$

Here A be the cross-section area of river, $C(x,t)$ be the concentration of pollutants, U be the water velocity in x- direction, D_{cx} be the dispersion coefficient of pollutants in the direction of x, Q be the rate of added along the river, $k$be the half saturated oxygen demand concentration for pollutant decay,$k_1$bethe rate of degradation for pollutant coefficient and X_o be the solvate oxygen concentration in water. We suppose that the river is assumed in uniform system. Therefore, we have supposed $A,\ U,\ D_{cx},\ k_1$as the constant parameters as constants over time and space. Therefore we suppose$D_{cx} = 0$ (for high pollutants) , $Q = 0$ and $k = 0$.

Using the above assumed conditions, the equation (1) becomes:

$$\frac{\partial(AC)}{\partial t} = -\frac{\partial(UAC)}{\partial x} - k_1 AC; \qquad 0 \le x < l, \quad t > 0 \qquad (2)$$

We suppose the following conditions to get the solution of the equation number (2):

$$C(x,0) = P_i \;\; ; \qquad x \ge 0\,, \qquad (3)$$

$$C(0,t) = \; R_o \;; \qquad t > 0, \qquad (4)$$

here P_i is the initial rate of pollution along the river and R_o is the pollutant rate at the origin.

Using the Laplace's transformation method in the equation numbers (2) and (4), we have

$$s\overline{C}(x,s) - C(x,0) = -U\frac{\partial\overline{c}(x,s)}{\partial x} - k_1\overline{C}(x,s); \qquad s > 0, \qquad (5)$$

$$\overline{C}(x,s) = \frac{R_o}{s}, \qquad (6)$$

where s is called the Laplace transform variable.

Now, applying the equation (3) in the equation (5), we have

$$s\overline{C}(x,s) - P = -U\frac{\partial\overline{C}(x,s)}{\partial x} - k_1\overline{C}(x,s);$$

Simplifying, we get

$$\frac{\partial\overline{C}(x,s)}{\partial x} + \left(\frac{s+k_1}{U}\right)\overline{C}(x,s) = \frac{P_i}{U}; \quad s > 0 \qquad (7)$$

Equation (7) is a linear differential equation in $\overline{C}$, so we have the integrating factor

$$I.F. = e^{\int\frac{k_1+s}{U}dx} == e^{\frac{k_1+s}{U}x}$$

Using this I.F. in equation (7) and then integrating, we get

$$\overline{C}(x,s)e^{\frac{k_1+s}{U}x} = \int\frac{P_i}{U}e^{\frac{k_1+s}{U}x}dx$$

$$= \frac{P_i}{U}\int e^{\frac{k_1+s}{U}x}dx$$

Simplifying the above equation, we get

$$\overline{C}(x,s) \;= \frac{P_i}{k_1+s} + c_1 e^{-\left(\frac{k_1+s}{U}\right)x} \qquad , \qquad (8)$$

where $c_1$is a constant. This constant is known as an arbitrary constant.

Now applying the condition (6) in the equation (8), we have

$$\frac{rR_o}{s} = \frac{P_i}{k_1+s} + c_1 e^{-\left(\frac{k_1+s}{U}\right).0}$$

that gives $c_1 = \frac{R_o}{s} - \frac{P_i}{k_1+s}$.

Using the above result for c_1 in the equation number (8), we have

$$\overline{C}(x,s) = \frac{P_i}{k_1+s} + \left(\frac{R_o}{s} - \frac{P_i}{k_1+s}\right) e^{-\left(\frac{k_1+s}{U}\right)x}$$

$$= \frac{P_i}{k_1+s} + \left(\frac{R_o}{s} - \frac{P_i}{k_1+s}\right) e^{-\left(\frac{k_1+s}{U}\right)x}$$

Therefore, we have

$$\overline{C}(x,s) = \frac{P_i}{(k_1+s)} + \frac{R_o}{s} e^{-\left(\frac{k_1+s}{U}\right)x} - \frac{P_i}{(k_1+s)} e^{-\left(\frac{k_1+s}{U}\right)x} \quad (9)$$

Now, we apply the inverse of Laplace transformation method to the above result (9), we get

$$C(x,t) = P_i e^{-k_1 t} + R_o \left\{ e^{-\frac{k_1}{U}x} . H\left(t - \frac{x}{U}\right) \right\} - P_i e^{-\left(\frac{k_1}{U}x + k_1 t\right)} . H\left(t - \frac{x}{U}\right) \quad (10)$$

where *H* is a Heaviside function which is defined as

$H\left(t - \frac{x}{U}\right) = 1, \quad if\ t - \frac{x}{U} > 0;$.

$H\left(t - \frac{x}{U}\right) = 0, \quad if\ 0 > t - \frac{x}{U};$

For $t > \frac{x}{U}$, the above equation (10) reduces to

$$C(x,t) = P_i e^{-k_1 t} + R_o e^{-\frac{k_1}{U}x} - P_i e^{-\left(\frac{k_1}{U}x + k_1 t\right)} \quad (11)$$

Now, we suppose the following quantities

$$x' = \frac{k_1}{U}x, \quad t' = k_1 t, \quad P_i' = P_i, \quad C'(x',t') = C(x,t), R_o' = R_o$$

Applying the above quantities in equation (11), we have

$$C'(x',t') = P_i' e^{-t'} + R_o' e^{-x'} - P_i' e^{-(t'+x')} \quad (12)$$

3. Results and discussion

Using the solution (12) of a dispersion problem in an unsteady case given by equation (2), we observed the concentration behavior of water pollutants. The concentration $C'(x,t)$ is measured in kg/m^3. We have applied most of the parametric values in the equation (12) as Pimpunchat et al. [11]:

$t' = 3.308(t = 0.4\ day),\ 4.962(t = 0.6\ day),\ 6.616(t = 0.8\ day),\ 8.27\ (t = 1.0 day),\ k_1 = 8.27\ per\ day,$

$P_i' = 0.04, \quad 0.06, \quad 0.08, \quad 0.10 \quad in \quad kg/m^3.$

$R_o' = 0.01, \quad 0.02, \quad 0.03, \quad 0.04 \quad in \quad kg/m^3.$

For different conditions and cases, we have shown the behavior of the concentration of pollutants profiles graphically.

Figure 1 shows concentration $C'(x,t)$ outline for the interval (0≤ $x' \leq 10$) with the constant time period ($t' = 3.308(t = 0.4\ day)$ and initial velocity $P_i{}'$as constant. Graphically we have found that as the distance x' increases concentration $C'(x,t)$ decreases. This approaches a fixed concentration close to the sink. In this case, nearby upstream the effect of time period is supreme and nearby downstream it is extremely small.

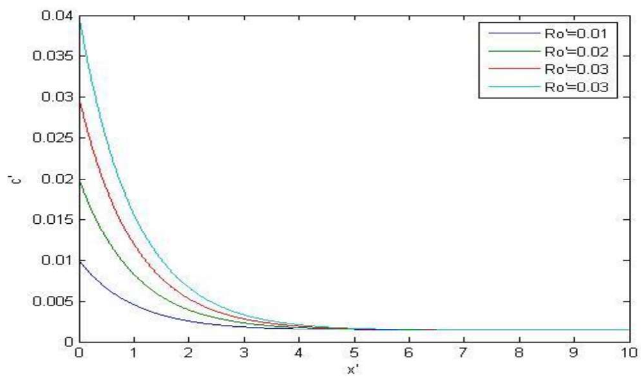

Fig. 1: Concentration profile in reference to various $R_o{}'$ for constant t'.

Figure 2 shows concentration $C'(x,t)$ for the interval (0≤ $x' \leq 10$)for various initial rate of pollutants $P_i{}'$ *and constant period of time* t'.We have seen that asthe distance x'increases, concentration $C'(x,t)$ decreases. Concentration $C'(x,t)$ approaches a fixed concentration close to the sink. In this case, we have observed that as the initial speed increases, the concentration $C'(x,t)$ also increases at all cross sectional area.

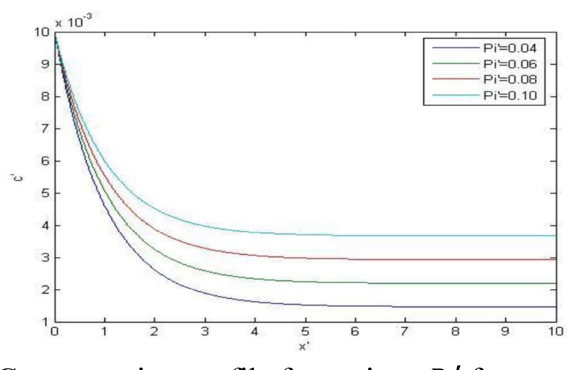

Fig. 2: Concentration profile for various $P_i{}'$ for constant t'.

Figure 3 shows concentration outline for the interval (0≤ $x' \leq 10$)for various period of time and initial speed as a constant $R_o{}' = 0.01$. From the graph, we have found as the distance x' increases, concentration $C'(x,t)$ decreases and the effect of period of time for pollutant concentration is almost negligible close to the upstream and extremely small close to the .downstream. It is further seen that as time period increases, concentration $C'(x,t)$also increases at all cross sectional area of the river.

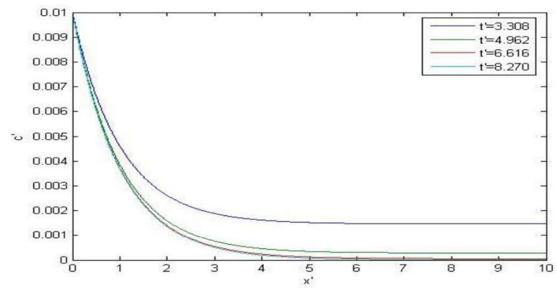

Fig. 3: Concentration profile for various t' and constant velocity

Figure 4 shows the concentration of pollutants profile versus the interval ($0 \leq x' \leq 10$) in reference to various period of time and various pollutant rate at the origin. From the graph, it is found that as the distance x' increases, concentration $C'(x,t)$ decreases and effect of period of time for pollutant concentration is almost negligible close to the upstream and supreme close to the downstream. It is further seen that as time period increases, concentration $C'(x,t)$ also increases at all cross sectional area of the river.

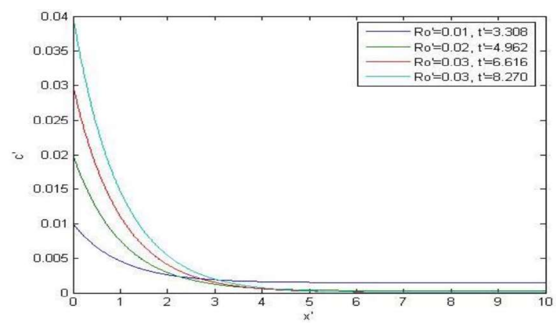

Fig. 4: Concentration profile for various $R_o{}'$ and t'

Figure 5 shows the concentration profile verses the distance ($0 \leq t' \leq 10$) for various value of initial rate of pollutants $R_o{}'$. From the graph, we have made a summary that as t'increases, concentration $C'(x,t)$ decreases and effect of period of time for pollutant concentration is little close to upstream and supreme close to downstream. It is further seen that as initial rate of pollutants increases, the concentration $C'(x,t)$ of pollutants also increases at any cross section of the river.

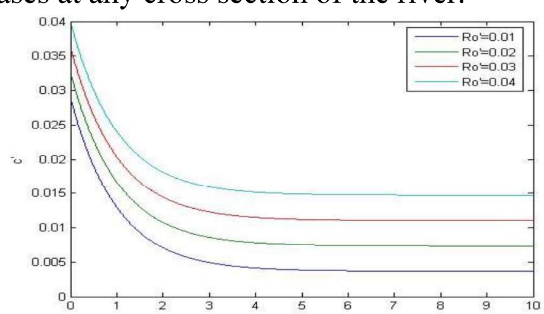

Fig. 5: Concentration profile versus time period t' for various $R_o{}'$

4. Conclusion

From this study, we have found that concentration of pollutants in river water is greater close to river water source and $C'(x,t)$ decline continuously as the distance from the source and time period growing. It is the general behavior of water pollutants with no added pollutants.

This study also summarizes that concentration $C'(x,t)$ versus x' in various time period t' with initial speed $P_i{}'$ decreases to many periods of time as x' growing. Very small concentration $C'(x,t)$ is observed close to downstream and as initial speed at origin growing, $C'(x,t)$ of water pollutants increases for all cross-section in river. The effect of period of time is supreme close to upstream; the effect of time period is high and very low near the downstream.

This model is important to those who want to study the concentration behavior of water pollutants under the different conditions for time and velocity along the river.

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