Enhancement of Generalized q-Holder's Integral Inequality on Finite Interval

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Abstract: q-calculus is the modern version of calculus without the notion of limits primarily introduced by Euler in eighteenth century but a systematic enhancement on it was done by F.H. Jackson early in the nineteenth century. The rapid growth on it is due to its application in physics and quantum computing.One of the applications of quantum calculus is the theory of integral inequality.In this paper, we aim at enhancing a generalized q-Holder's integral inequality on a finite interval.

Key-Words: Quantum Calculus, Holder's inequality, q-Holder's Inequality

1. Introduction

The investigation of calculus without the notion of limits is quantum calculus or q-calculus. In q-calculus, the mathematical relations are converted into q framework which can be recaptured in the classical framework when $q \to 1$. The rapid growth on q-calculus is due to its applications in various branches of mathematics and physics such as number theory, orthogonal polynomials, basic hyper-geometric series, calculus of variations, quantum computing, quantum theory, mechanics, theory of relativity and so on.

The first formula in q-calculus was obtained by Leonhard Euler in the middle of the eighteenth century. But the systematic initiation on q-calculus was done by F.H. Jackson by introducing the notion of q-derivative and definite q-integral in 1908 and 1910 respectively. In the second half of the twentieth century, there was a significant increase in the area of q-calculus due to its application in physics and mathematics.One of the applications of quantum calculus is in the theory of integral inequality.

This paper aims at enhancing q-analogue of the generalized q-Holder's integral inequality on a finite interval.

2.Preliminaries and Results

F.H. Jackson [3] reintroduced the Euler-Jackson q-difference operator, called q-derivative which is defined as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, t \in (0, b), q \in C \backslash \{1\}$$

for $f : [0, b) \to \mathbb{R}$, $0 \leq b < \infty$.

It is clear that if $f(x)$ is differentiable, then $\lim_{q \to 1} D_q f(t) = \frac{df(t)}{dt}$. F.H. Jackson introduced the concept of q-definite integral extending the idea of q-calculus. The q-definite integral is defined as

$$\int_0^x f(t) d_q t = (1-q) x \Sigma_{n=0}^{\infty} q^n f(q^n x), x \in (0, \infty)$$

Tariboon et al. in 2013 [2] defined q-derivative on finite interval as follows:

Definition 2.1 The q-derivative of a continuous function $f : J = [a, b] \subset \mathbb{R} \to \mathbb{R}$ at a point $t \in J$ on [a,b] is defined as

$$_aD_q f(t) = \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)}, t \neq a$$

and

$aD_q f(a) = \lim_{t \to a} aD_q f(t)$

Tariboon et al. [2] defined q-integral on finite interval J =[a,b] as follows:

Definition 2.2 Assume $f : J \to \mathbb{R}$ is a continuous function. Then the q-integral on J is defined by

$$\int_a^x f(t)_a d_q t = (1-q)(x-a)\Sigma_{n=0}^{\infty} q^n f(q^n + (1-q^n)a).$$

Definition 2.3 Let a and b be vectors in R^n such that $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ where a_i and b_i are real numbers.Then the following result is credited to Otto Holder, known as Holder's inequality in discrete form.

Theorem 1 If $p > 1$ and $q > 1$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

then

$$\Sigma_{i=1}^{n}|a_i b_i| \le (\Sigma_{i=1}^{n}|a_i|^p)^{\frac{1}{p}}(\Sigma_{i=1}^{n}|b_i|^q)^{\frac{1}{q}}.$$

It is noted that if $n = 2$, and $p = 2 = q$, then the inequality (6) reduces to the famous Cauchy - Schwarz inequality.

Theorem 2 [1] Suppose that p and q are real numbers with $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and f and g are integrable functions. Then the functions $|f|^p$ and $|g|^q$ are also integrable and the following Holder's inequality holds:

$$\int_a^b |f(t)g(t)|dt = (\int_a^b |f(t)^p|dt)^{\frac{1}{p}})(\int_a^b |g(t)|^q dt)^{\frac{1}{q}})$$

In order to derive the generalized Holder's inequality we need the help of following two lemmas known as Young's inequality.

Lemma 1. For $a, b \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

is satisfied.

This inequality can be generalized as follows:

lemma 2. For $i = 1, 2, \ldots, n$, let $x_i \geq 0$ and $\alpha_i > 1$ such that $\Sigma_{i=1}^{n} \frac{1}{\alpha_i} = 1$, then the inequality

$$\prod_{i=1}^{n} Q_i^{\frac{1}{\alpha_i}} \leq \Sigma_{i=1}^{n} \frac{Q_i}{\alpha_i}$$

is valid.

Tariboon et al [2] stated and proved the q-Holder's integral inequality on a finite interval $J = [a, b]$ as follows:

Theorem 3 Let $x \in J, 0 < q < 1, p_1, p_2 > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$, then

$$\int_a^x |f(t)g(t)|_a d_q t \leq \left(\int_a^x |f(t)|^{p_1} a d_q t\right)^{\frac{1}{p_1}} \left(\int_a^x |g(t)|^{p_2} a d_q t\right)^{\frac{1}{p_2}}.$$

The generalized q-Holder's integral inequality is stated as follows:

Theorem 4. Let $f_1, f_2, ..., f_n$ be functions such that the integral exist. Then the inequality

$$\int_0^a |\prod_{i=1}^m f_i(t)| d_q t \leq \prod_{i=1}^m (\int_0^a |f_i(t)|^{\alpha_i} d_q t)^{\frac{1}{\alpha_i}}$$

holds for $\alpha_i > 1$ such that $\Sigma_{i=1}^m \frac{1}{\alpha_i} = 1$

Proof: Let, $\alpha_i > 1$ such that $\Sigma_{i=1}^m \frac{1}{\alpha_i} = 1$. Then by definition of q-Jackson integral we obtain,

$$\int_0^a f(t) d_q t = (1-q) a \Sigma_{k=0}^{\infty} q^k f(aq^k)$$

And from the relations

$$\Sigma_{k=1}^n |\prod_{i=1}^m Q_{i,k}| \leq \prod_{k=1}^m (\Sigma_{i=1}^n)(|Q_{i,k}|^{\alpha_i})^{\frac{1}{\alpha_i}}$$

we obtain

$$\int_0^a |\prod_{i=1}^n f_i(t)| d_q t \ = \ (1-q) a \Sigma_{k=0}^{\infty} \prod_{i=1}^n |f_i(aq^k)| q^k.$$

$$
\begin{aligned}
&\leq (1-q)a\prod_{i=1}^{n}(\Sigma_{k=0}^{\infty}|f_i(aq^k)|^{\alpha_i}q^k)^{\frac{1}{\alpha_i}}.\\
&= ((1-q)a)^{\sum_{i=1}^{n}\frac{1}{\alpha_i}}\prod_{i=1}^{n}(\Sigma_{k=0}^{\infty}|f_i(aq^k)|^{\alpha_i}q^k)^{\frac{1}{\alpha_i}}.\\
&= \prod_{i=1}^{n}((1-q)a)^{\frac{1}{\alpha_i}}\prod_{i=1}^{n}(\Sigma_{k=0}^{\infty}|f_i(aq^k)|^{\alpha_i}q^k)^{\frac{1}{\alpha_i}}.\\
&= \prod_{i=1}^{n}((1-q)a\Sigma_{k=0}^{\infty}|f_i(aq^k)|^{\alpha_i}q^k)^{\frac{1}{\alpha_i}}.\\
&= \prod_{i=1}^{n}(\int_0^a|f_i(t)|^{\alpha_i}d_qt)^{\frac{1}{\alpha_i}}
\end{aligned}
$$

This completes the proof.

Now, we proceed to prove the generalized q-Hölder integral inequality on a finite interval.

3. Main Result

Theorem 5 Let $J=[t_k,t_{k+1}]=[a,b]$, let $x\in J, 0<q<1$, and $\alpha_i>1$, such that $\Sigma_{i=1}^{n}\frac{1}{\alpha_i}=1$. Let, $f_1,f_2,\ldots,f_n$ be functions such that their integral exist, then the inequality

$$\int_a^x|\prod_{i=1}^{n}f_i(t)|_a d_qt\leq\prod_{i=1}^{n}(\int_a^x|f_i(t)|_a^{\alpha_i}d_qt)^{\frac{1}{\alpha_i}}$$

holds.

Proof Let $J=[t_k,t_{k+1}]=[a,b]$ Replace $t_k=a, t_{k+1}=b$ and $q_k=q$.

Then,

$$\int_a^x |\prod_{i=1}^n f_i(t)|_a d_q t = (1-q)(x-a)\Sigma_{k=0}^{\infty} \prod_{i=1}^n |f_i(q^k x + (1-q^k a))|q^k.$$

$$= (1-q)(x-a)\prod_{i=1}^n (\Sigma_{k=0}^{\infty} |f_i(q^k x + (1-q^k)a)|^{p_i} q^k)^{\frac{1}{p_i}}.$$

$$= ((1-q)(x-a))^{\Sigma_{i=1}^n} \prod_{i=1}^n (\Sigma_{k=0}^{\infty} |f_i(q^k x + (1-q^k)a|^{p_i} q^k)^{\frac{1}{p_i}}.$$

$$= \prod_{i=1}^n ((1-q)(x-a))^{\frac{1}{p_i}} \prod_{i=1}^n (\Sigma_{k=0}^{\infty} |f_i(q^k x + (1-q^k)a)|^{p_i} q^k)^{\frac{1}{p_i}}.$$

$$= \prod_{i=1}^n ((1-q)(x-a)\Sigma_{k=0}^{\infty} |f_i(q^k x + (1-q^k)a|^{p_i} q^k)^{\frac{1}{p_i}}.$$

$$= \prod_{i=1}^n (\int_a^x |f_i(t)|_a^{p_i} d_q t)^{\frac{1}{p_i}}.$$

This completes the proof.

References

[1] Nantomach, Kwara,(2017), Generalized Holder's and Minkowski's Inequalities for Jackson's q-integral and Some Applications to the Incomplete q-gamma functions, *Abstract, Applied and Analysis, Hindwi.*

[2] Tariboon, J. and Ntouyas, S.K., (2014), Quantum Inequalities on finite intervals, *Journal of Inequalities and Applications 2014(121).*

[3] Kac,V. and Cheung, P.,(2002) Quantum Calculus, *Springer, New York.*