

PRELIMINARY CONCEPT AND RESULT OF FUZZY METRIC SPACE IN POINT SET TOPOLOGY

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ABSTRACT

The innovative concept of fuzzy mathematics has become one of the interesting areas of research since last fifty-five years, which was first introduced by Zadeh in 1965. Many researchers connected the fuzzy concept in different forms of metric spaces. The results about the point sets have discussed, mainly topological properties of sets, connecting with the fuzzy metric space. The propose of this paper is to study the point set topology in fuzzy metric space, especially by introducing the concept of open and closed balls and discuss some of the common properties. Moreover, we introduce the concept of compactness and pre-compactness in fuzzy metric space.

Keywords: Closed set, Compactness, Fuzzy metric space, Induced topology, Open set.

INTRODUCTION

A. L. Zadeh (1965) proposed the fuzzy set concept, which was a brand-new idea in mathematics. This serves as the cornerstone for removing the ambiguity from our day-to-day lives. As a result, numerous academics in a variety of domains, including artificial intelligence, computer science, and mathematics, have extensively expanded the idea of fuzzy sets and fuzzy logic. Which is also used in mathematics, engineering, game theory, and optimization theory. Additionally, they introduced the novel concept of the space concern with fuzzy and used several forms of the fundamental topology of fuzzy sets. In this study, we employ the fuzzy metric space notations developed by George and Veeramani (1994), which are a modified version of the fuzzy metric space symbols

previously investigated by Kramosil and Michalek (1975). Which has a wide range of analytical applications and outcomes, and there are numerous notations and findings derived from classical metric spaces that can be expanded upon and applied to the study of fuzzy metric spaces. Additionally, Grabiec (1988) demonstrated how the principle of contraction in fuzzy metric spaces works.

By utilizing the idea of continuous t-norm, George et al. (1994) provided the modified notation of fuzzy metric spaces. In this article, we adapted the fuzzy metric space notions utilized by Kramosil and Michalek (1975) and provided a formulation of Hausdroff topology that included the fuzzy metric space concept. We demonstrate that each metric results in a fuzzy metric. We also demonstrate that any separable fuzzy metric space has a pre compact fuzzy metric and that a fuzzy metric space is compact if it is pre compact and complete, demonstrating that the topology generated by the fuzzy metric space is metrizable. By utilizing the idea of fuzzy metric spaces, George et al. (1994) provided the modified notation of fuzzy metric spaces. Additionally, we define the Hausdroff topology on fuzzy metric space, which was first utilized by Kramosil and Michalak (1975), and we demonstrate some established findings about the connections between fuzzy metric space with open ball and closed ball.

There were still numerous expansions of the terms for metric and metric space, in other words, fuzzy metric spaces. Bakhtin (1989) and Czerwik (1993) offered the new conceptions about a space where the triangle inequality, a generic condition was applied, with the basis of generalized version of the Banach Contraction principle (1922).

PRELIMINARIES

Here we will introduce the following definitions which are frequently used in our work.

Triangular norm: A mapping $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is said to be continuous triangular norm (t-norm) if for all $m, n, o, p \in [0,1]$, the following properties are satisfied.

- (i) $m * 1 = m$
- (ii) $m * n = n * m$
- (iii) If $m \leq o$ and $n \leq p$ then $p * n \leq o * p$
- (iv) $m * (n * o) = (m * n) * o$
- (v) $*$ is continuous.

Example: As an example of continuous t-norm, the usual product, in which the binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is defined by $a * b = ab$, then it is clear that $*$ is commutative and associative.

Now, $f(a, b) = a * b = ab$, for all $a, b \in [0,1]$

Let $A = [a, b] \subseteq [0,1]$

Hence $f^{-1}(A) = [c, d] \subseteq [0,1]$

Then $a * b$ is continuous.

Also, $a * 1 = a$, for all $a \in [0,1]$.

Finally,

$a * b = ab$, if $a \leq c$, $b \leq d$ and $cd = c * d$, for all $a, b, c, d \in [0,1]$.

Therefore, $a * b = ab$ is continuous t-norm.

Moreover, $T_m(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ are the examples of continuous t-norms.

Fuzzy metric space: An ordered triple $(U, N, *)$ is called to form a fuzzy metric space if U is an arbitrary set, $*$ is a continuous t-norm and N is a fuzzy set on $U^2 \times (0,1)$ satisfying the following conditions:

- (i) $N(u, v, t) > 0$,
- (ii) $N(u, v, t) = 1$ for all $t > 0$ if and only if $u = v$
- (iii) $N(u, v, t) = N(v, u, t)$,
- (iv) $N(u, v, t) * N(v, w, s) \leq N(u, w, t + s)$ for all $t, s > 0$,
- (v) $N(u, v, \cdot): (0, \infty) \rightarrow [0,1]$ is continuous

Where $N(u, v, t)$ is the degree of nearness between u and v in the basis of $t > 0$.

Example: Let $U = R$ assume that $a * b = ab$ and

$$N(u, v, t) = \left[\exp \cdot \left(\frac{|u - v|}{t} \right) \right]^{-1}$$

for all $u, v \in U$ and $t \in (0, \infty)$. Then $(U, N, *)$ is a fuzzy metric space.

Proof: clearly,

- (i) $U(u, v, t) > 0$ and
- (ii) $U(u, v, t) = 1$ if and only if $u = v$.
- (iii) In order to prove

$$N(u, v, t) \cdot N(v, w, s) \leq M(u, w, t + s)$$

as we have,

$$|u - w| \leq \left[\frac{t + s}{t} \right] |u - v| + \left[\frac{t + s}{s} \right] |v - w|$$

i.e.

$$\frac{|u - w|}{t + s} \leq \frac{|u - v|}{t} + \frac{|v - w|}{s}$$

$\exp. \left[\frac{|u-w|}{t+s} \right] \leq \exp. \left[\frac{|u-v|}{t} \right] \exp. \left[\frac{|v-w|}{s} \right]$ thus

$$N(u, v, t) \cdot N(v, w, s) \leq N(u, w, t + s).$$

(iv) Let us consider a sequence $\{t_n\} \in [0, \infty)$ such that the sequence $\{t_n\}$ converges to $t \in [0, \infty)$ then

$$\lim_{n \rightarrow \infty} |t_n - t| = 0.$$

Let us assume that for $u, v \in U$. Since the function e^u is continuous on R we have $e^{\frac{|u-v|}{t_n}}$ converges to $e^{\frac{|u-v|}{t}}$ as t_n converges to t , with respect to the usual metric. Therefore

$$(U, N, *): (0, \infty) \rightarrow [0, 1]$$

is continuous. i. e. $N(u, v, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous. Thus $(U, N, *)$ is a fuzzy metric space.

RESULTS AND DISCUSSION

Open ball: Let $(U, N, *)$ is a fuzzy metric space consider an open ball $B(u, r, t)$ with centre $x \in U$ and radius r such that $0 < r < 1$ and > 0 , we have

$$B(u, r, t) = \{v \in U: N(u, v, t) > 1 - r\}$$

F-bounded: Let $(U, N, *)$ be a fuzzy metric space. A subset A of U is said to be F -bounded if and only if there exists $t > 0$ and $0 < r < 1$ such that

$$N(u, v, t) > 1 - r$$

for all $u, v \in A$.

Remark: If $(U, N, *)$ be a fuzzy metric space induced by a metric d on U . Then $A \subseteq U$ is F -bounded iff it is bounded.

Convergent sequence: A sequence $\{x_n\}$ in a fuzzy metric space $(U, N, *)$ is said to be convergent to $u \in U$ if $\lim_{n \rightarrow \infty} N(x_n, u, t) = 1$ for all $t > 0$.

And is Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$ there exists $n_0 \in N$ such that for all $m, n \geq n_0$ we have

$$N(x_n, x_m, t) > 1 - \epsilon$$

Closed ball: Suppose $(U, N, *)$ is a fuzzy metric space. Now, a closed ball is defined as having a centre $u \in U$ and radius r such that $0 < r < 1$ and $t > 0$ as

$$B[u, r, t] = \{v \in U; N(u, v, t) \geq 1 - r\}.$$

Now we discuss some theorems as follows:

Theorem: Every open ball in a fuzzy metric space $(U, N, *)$ is open set.

Proof: Let us assume that an open ball $B(u, r, t)$. Now $v \in B(u, r, t) \Rightarrow N(u, v, t) > 1 - r$ since $N(u, v, t) > 1 - r$

we can find a $t_0; 0 < t_0 < t$ such that $N(u, v, t_0) > 1 - r$

since

also we have

$$r_0 > 1 - s > 1 - r$$

now for a given r_0 and s such that $r_0 > 1 - s$, we can find $r_1; 0 < r_1 < 1$ such that

$$r_0 * r_1 \geq 1 - s$$

If we consider the ball $B(v, 1 - r_1, t - t_0)$. We claim

$$B(v, 1 - r_0, t - t_0) \subset B(u, r, t)$$

Now

$$w \in B(v, 1 - r_1, t - t_0) \Rightarrow N(v, w, t - t_0) > r_1$$

Therefore

$$\begin{aligned} N(u, w, t) &\geq N(u, v, t_1) * N(v, w, t - t_0) \\ &\geq r_0 * r_1 \geq 1 - s > 1 - r \end{aligned}$$

Therefore $w \in B(u, r, t)$ and hence

$$B(v, 1 - r_1, t - t_0) \subset B(u, r, t).$$

Thus every open ball is open set.

Theorem: Every fuzzy metric space is Hausdroff.

Proof: Let $(U, N, *)$ be a fuzzy metric space.

In order to complete the proof we will show that distinct points have disjoint neighborhoods.

Let u, v be any two point of U , such that $u \neq v$ then

$$0 < N(u, v, t) < 1$$

Let $N(u, v, t) = r$; for some r where;

$$0 < r < 1.$$

for each $r_0; r < r_0 < 1$;

We can find a r_1 such that

$$r_1 * r_0 \geq r_0$$

let us assume any two open balls

$$B\left(u, 1 - r, \frac{1}{2}t\right)$$

and $B\left(v, 1 - r, \frac{1}{2}t\right)$

Clearly

$$B\left(u, 1 - r_1, \frac{1}{2}t\right) \cap B\left(v, 1 - r_1, \frac{1}{2}t\right) = \phi$$

If there exists

$$w \in B\left(u, 1 - r_1, \frac{1}{2}t\right) \cap B\left(v, 1 - r_1, \frac{1}{2}t\right)$$

$$\begin{aligned} \text{Then } r = N(u, v, t) &\geq N\left(u, w, \frac{1}{2}t\right) * N\left(w, v, \frac{1}{2}t\right) \\ &\geq r_1 * r_1 \geq r_0 > r. \end{aligned}$$

That contradicts our assumption.

Hence $(U, N, *)$ is Hausdroff.

Theorem: Let $(U, N, *)$ be a fuzzy metric space and T be the topology induced by the fuzzy metric. Then for a sequence $\{u_n\}$ in U , $u_n \rightarrow u$ if and only if

$$N(u_n, u, t) \rightarrow 1 \text{ as } n \rightarrow \infty$$

Proof : Let us assume $t > 0$.

Suppose $u_n \rightarrow u$. Then for $0 < r < 1$ then there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(u, r, t)$ for all $n \geq n_0$.

It follows that

$$N(u_n, u, t) > 1 - r$$

So

$$1 - N(u_n, u, t) < r$$

Hence

$$N(u_n, u, t) \rightarrow 1 \text{ as } n \rightarrow \infty$$

Conversely, if for each $t > 0$ $N(u_n, u, t) \rightarrow 1$ as $n \rightarrow \infty$ then for $0 < r < 1$, there is $n_0 \in \mathbb{N}$ which gives $1 - N(u_n, u, t) < r$ for all $n \geq n_0$

thus $u_n \in B(u, r, t)$ for all $n \geq n_0$

and hence $u_n \rightarrow u$.

Theorem: Every closed ball in a fuzzy metric space $(U, N, *)$ is closed set.

Proof: Let $v \in \overline{B(u, r, t)}$. As U is countable, so there exists a sequence $\{v_n\}$ in $B[u, r, t]$ such that

$$v_n \rightarrow v$$

So , $N(v_n, v, t) \rightarrow 1$ for all $t > 0$.

Now for a given $\epsilon > 0$ we have

$$N(u, v, t + \epsilon) \geq N(u, v_n, t) * N(v_n, v, t)$$

Hence, $N(u, v, t + \epsilon) \geq \lim_{n \rightarrow \infty} N(u, v_n, t) * \lim_{n \rightarrow \infty} (v_n, v_1, t)$

$$\geq (1 - r) * 1 = 1 - r$$

If $N(u, v_n, t)$ is bounded, so the sequence $\{v_n\}$ has a subsequence which is denoted by $\{v_{n_k}\}$ for which

$$\lim_{n \rightarrow \infty} N(u, v_{n_k}, t)$$

exists.

In particular, for $n \in \mathbb{N}$, take $\epsilon = \frac{1}{n}$ then $N(u, v, t + \frac{1}{n}) \geq 1 - r$

hence,

$$N(u, v, t) = \lim_{n \rightarrow \infty} N(u, v, t + \frac{1}{n}) \geq 1 - r$$

Thus $v \in B[u, r, t]$

Therefore $B[u, r, t]$ is a closed set.

Pre-compact: A fuzzy metric space $(U, N, *)$ is called pre-compact if for each r , with $0 < r < 1$ and each $t > 0$ there exists a finite subset A of U , where

$$U = \cup_{a \in A} B(a, r, t).$$

In such condition we can say that M is a pre-compact fuzzy metric space on U . A fuzzy metric space $(U, N, *)$ is known as compact if $(U, N, *)$ is compact topological space.

Theorem: A fuzzy metric space is compact if and only if it is pre-compact and complete.

Proof: Let us assume that $(U, N, *)$ is a complete fuzzy metric space for each r ; with $0 < r < 1$ and each $t > 0$ the open cover $\{B(u, r, t): u \in U\}$ of U has a finite subcover. Hence $(U, N, *)$ is pre-compact. On the other hand, every Cauchy sequence $\{u_n\}$ for $n \in \mathbb{N}$ in $(U, N, *)$ has a limit point $v \in U$.

Also $(U, N, *)$ be a fuzzy metric space. If a Cauchy sequence has a limit to a point $u \in U$, then the sequence converges to u . Then $\{u_n\}$ converges to v , thus $(U, N, *)$ is complete.

Conversely, let us assume that $\{u_n\}$ be a sequence in U as we know that the completeness of $(U, N, *)$ follows that $\{u_n\}$ has a cluster point. As $(U, N, *)$ is metrizable and every sequentially compact metrizable space is compact and hence we conclude that $(U, N, *)$ is compact.

CONCLUSION

This study generalized the concept of fuzzy metric space in the sense of George and Veeramani by presenting the definition of fuzzy metric space and point set topology. The proof of some known results of point set topology connecting with fuzzy metric space are discussed. The common properties of point set topology in fuzzy metric space are discussed. At last, we discussed the concept of compactness and pre-compactness in fuzzy metric space by using some properties.

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