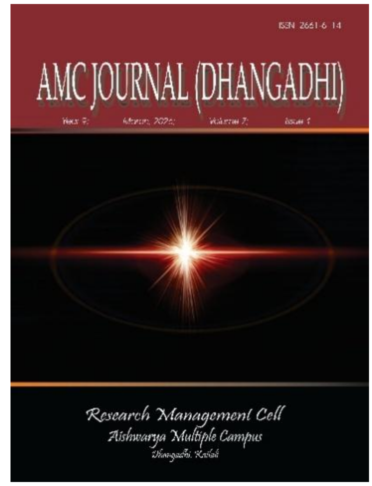
$\alpha - \varphi$ Contractions in Fractal Spaces



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Abstract

The aim of this work is to introduce α-admissible mapping in fractal spaces and prove fixed point theorem in the metric space of fractals. To establish the theorem we use $\alpha - \varphi$ contraction condition on α-admissible mapping. Here we consider the underlying metric space (X, d) is complete metric space and by using previous knowledge the fractal space on this underlying space (X, d) is also complete metric space. This completeness property of fractal space yields the existence of fixed point of invariant mapping defined on fractal space. To clarify the convergence of fractals we introduce matrix of convergent sequences of contractions.

Keywords: Fractal space, Hausdorff distance of sets**,** $\alpha - \varphi$ contraction, α-admisible.

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Introduction and preliminaries

Let (X, d) be a complete metric space and $K(X)$ be class of compact sub-sets of X, then we define following definitions;

Definition 1: (Hutchinson, 1981) For any $A \in K(X)$ and $x \in X$, the distance $d(x, A) = min.\{\, d(x, y)\ \forall\ y \in A\}$

Definition 2: (Hutchinson, 1981)The distance of any two compact sets $A, B \in K(X)$ is given by $d(A, B) = max.\{d(x, B) \forall x \in A\,\}$ and $d(B, A) = max.\{d(y, A) \forall y \in B\,\}$

Definition 3: (Hutchinson, 1981) $H_d = max.\{d(A, B)\,, d(B, A)\}$ is called Hausdorff distance of any two compact sub-set of X.

Definition 4: (Hutchinson, 1981) Let (X, d) be a complete metric space and $\mathcal{F} = \{f_1\,, f_2, f_3 \dots \dots \dots f_n\}$ be finite family of self-mappings in(X, d), with contractivity factors $\alpha_1, \alpha_2, \alpha_3 \dots \dots \dots .. \alpha_n$ then the space $\{(X, d), f_1\,, f_2, f_3 \dots \dots \dots f_n\}$ is called hyperbolic iterated function system and denoted by IFS.

Definition 5: (Hutchinson, 1981)The set to set mapping $\mathcal{F}: K(X) \to K(X)$ is defined by $\mathcal{F}(A) = \bigcup_{i=1}^{n} f_i(A)$ where $\{i = 1,2,3 \dots . n\}$for all $A \in K(X)$ is called invariant operator.

Hutchinson (Hutchinson, 1981) first states that The space $K(X)$ with Hausdorff distance $H_d$is complete metric space if the given metric space (X,d) is complete and it is denoted by$(K(X),H_d)$. Moreover he state that the invariant operator $\mathcal{F}(A)=\bigcup_{i=1}^{n} f_i(A)\ \forall\ A\in K(X)$ is contraction in $(K(X),H_d)$. with contractivity factor $\alpha = max.\{\alpha_1,\alpha_2,\alpha_3 \ldots\ldots \alpha_n\}$. Where α_i is contraction factor of f_i for $\{i=1,2,3 \ldots\ldots n\}$. Thus by Banach fixed point theorem (Banach, 1922) there exist a unique $C\in K(X)$ such that $\mathcal{F}(C)=C$.

For the detail review of fractals and fractal space, see(Barnsley, 1993). In the view of Barnsley [3, p. 82], we can define following definition and theorems.

Theorem 1: Let $\{(X,d); f_1, f_2, f_3 \ldots\ldots\ldots f_n\}$ be hyperbolic iterated function system and $(K(X),H_d)$ is induced Hausdorff metric space of compact sub-sets of X. The set to set mapping $\mathcal{F}: K(X)\to K(X)$ is defined by $\mathcal{F}(A)=\bigcup_{i=1}^{n} f_i(A)$ for all $A\in K(X)$ is contraction mapping on the complete metric space, $(K(X),H_d)$, with contractivity factor $\alpha = max.\{\alpha_1,\alpha_2,\alpha_3 \ldots\ldots \alpha_n\}$.That is $H_d(\mathcal{F}(A),\mathcal{F}(B)\leq \alpha H_d(A,B)\forall\ A,B\ \in K(X)$. By Banach contraction theorem there exists unique fixed set $C\in K(X)$ such that $\mathcal{F}(C)=\bigcup_{i=1}^{n} f_i(C)=C$.This fixed compact set is called attractor of IFS or fractal. The infinite iteration of $\mathcal{F}$ converges to C.

i.e. for any $A\in K(X), then\ \lim_{n\to\infty}\mathcal{F}^n(A)=C$.

Song -il Ri (Ri, 2016) applied the improved concept of φ-contraction in fractal space and proved the following fixed point theorem;

Theorem 2: Let a function $\varphi:[0,+\infty)\to[0,+\infty)$ with $\varphi(t)<t, \limsup_{c\to t^+}.\varphi(c)<t$ for $t>0$ and $inf.\{\limsup_{c\to t^-}.\varphi(c)=t\}>0$.Here φ may not be increasing and upper semi-continuous from right. Let (X,d) be complete metric space and self- map $\mathcal{F}=\{f_1, f_2, f_3 \ldots\ldots\ldots f_n\}$ be φ –contractive. Then $\mathcal{F}: K(X)\to K(X)$ is also φ –contractive and has unique fixed point $C\in K(X)$. More over for any $A\in K(X)$, $\lim_{n\to\infty}\mathcal{F}^n(A)\to C$ in $(K(X),H_d)$.

The extension of Hutchison's theory to semi-metric space is introduced by Bessenyei and Penzes(Bessenyei & Pénzes, 2022) in 2022.If (X,d) be any complete semi-metric space with triangle function $\emptyset$,continuous at origin. Then the fractal metric space $(K(X),H_d)$ induced by (X,d) is also semi-metric space with the same triangle function $\emptyset$. The triangle function $\emptyset: R_+^2\to R_+$ is generalization of triangle inequality of ordinary metric d in any set X. The same concept was used by Kocsic and Pales (Kocsis & Páles, 2022).They proved the following fixed point theorem in complete semi-metric space $(K(X),H_d)$ using the result of Bessenyei and Pales (Bessenyei & Páles, 2017) in 2014.

Theorem 3: Let (X,d) be a complete semi-metric space with an upper semi-continuous triangle function $\emptyset: R_+^2\to R_+$ for d , $\varphi: R^+\to R^+$ be a comparison function and let $f_1, f_2, f_3 \ldots\ldots\ldots f_n$ be φ −contraction in X.Then $\mathcal{F}$ is also φ −contraction and there exist unique fractal in $K(X)$ with respect to system $\{f_1, f_2, f_3 \ldots\ldots\ldots f_n\}$.

There are many results are published for the fixed point theorems in semi-metric spaces. Fixed point theorems in complete semi-metric spaces were established by Bessenyei and P´ales (Bessenyei &

Páles, 2017) and also by Jachymski, Matkowski, and Swiatkowski,(Jachymski et al., 1995) which generalized the basic results of Browder (Browder, 1979) and Matkowski ,(Matkowski, 1975) and which enjoy many extensions such as Miculescu and Mihail (Miculescu & Mihail, 2017), Mitrovi´c and Hussain (Mitrovic & Hussain, 2019).

Definition 6: (Hutchinson, 1981) The fixed point of invariant operator $\mathcal{F}: K(X) \to K(X)$ is called attractor of $\mathcal{F}$.In other words, $C \in K(X)$, is called attraction if, $\mathcal{F}(C) = \bigcup_{i=1}^{n} f_i(C) = C$.

The main goal of this paper is to prove that the invariant operator $\mathcal{F}$ satisfy $\alpha - \varphi$ contraction in $(K(X), H_d)$. Also we will prove the existence of unique fractal C with respect to this IFS We use the result of B. Samet et al(Samet et al., 2012). and N Shahzad et al.(Shahzad et al., 2016) to prove the main result.

Convergence of fractal

The convergence sequence of fractals proved by Hutchinson (Hutchinson, 1981) and Minirani S, Sunil Mathew (S & Sunil Mathew, 1914) is not clear and not simple so we present convergence of fractals in a scientific manner here;

Let $\{(X,d), f_1, f_2, f_3 \dots\dots\dots f_m\}$ be hyperbolic iterated function system with attractor K and $(K(X), H_d)$ is induced Hausdorff metric space of compact sub-sets of X.Let $\{f_{nj}\}$ be sequences of contraction mappings converging to f_j ; $(j = 1,2, \dots . m)$ point wise in X. These sequences are shown following matrix form of iterated function system of complete metric space(X, d) .

$$\begin{bmatrix} f_{11}, f_{12}, f_{13} \dots \dots . . f_{1m} \\ f_{21}, f_{22}, f_{23} \dots \dots . . f_{2m} \\ \dots \dots \dots \dots \dots \dots . \dots \dots . . \\ \downarrow \quad \downarrow \quad \downarrow \cdots . \dots \dots . . \downarrow \\ f_1, f_2, f_3 \dots . \dots \dots . . f_m \end{bmatrix} \; attractors \begin{bmatrix} K_1 \\ K_2 \\ . . \\ \downarrow \\ K \end{bmatrix} \; invarient\ mappings \begin{bmatrix} F_1 \\ F_2 \\ . . \\ \downarrow \\ F \end{bmatrix}$$

Here each column shows converging sequence $\{f_{nj}\} \to f_j$. Every row is IFS with attractor K_n. Since sequences $\{f_{nj}\} \to f_j$ then sequence $\{K_n\} \to K$ as $n \to \infty$. The above statements are valid because according to Hutchinson (Hutchinson, 1981) to every finite set of contraction mappings in complete metric spaces (X, d) there exists unique compact sub-set of X as a fixed point in $(K(X), H_d)$.The most interesting information given by above matrix is that "if $\{F_n\}$ be sequence of invariant mappings from $K(X) \to K(X)$ defined by $\mathcal{F}_n(A) = \bigcup_{i=1}^{n} f_{nj}(A)$ in each row, then this sequence converges to $\mathcal{F}$ the invariant map induced by $f_1, f_2, f_3 \dots\dots\dots f_m$."

Main Results

In this section first we prove the invariant mapping $\mathcal{F}: K(X) \to K(X)$ is α-admissible then $\mathcal{F}$ satisfy $\alpha - \varphi$ contraction in $(K(X), H_d)$ and finally we prove fixed point theorem of $\mathcal{F}$.

$\mathcal{F}$ is α-admissible

Definition 7: Let f be a self- mapping in a complete metric space (X, d) and $\alpha: X \times X \to [0, \infty)$.The map f is called α-admissible if $\alpha(x, y) \geq 1 \Rightarrow \alpha(f(x), f(y)) \geq 1$

Now we show that this relation is true for complete metric space $(K(X), H_d)$

Definition 8: Corresponding to a mapping $\alpha: X \times X \to [0, \infty)$, we can define $\alpha: K(X) \times K(X) \to [0, \infty)$ by;

$\alpha(A, B) = min.\{\alpha(x, y) \; \forall \; x \in A \; and \; y \in B\}$ .For $A, B \in K(X)$.

If $\alpha(x, y) \geq 1 \;\; \forall \; x \in A \; and \; y \in B \; then \; \alpha(A, B) \geq 1$

On the basis of this definition let us define α-admissible mapping $\mathcal{F}: K(X) \to K(X)$ which is defined by $\mathcal{F}(A) = \bigcup_{i=1}^{n} f_i(A)$ for all $A \in K(X)$ where f_i ($i = 1,2,3 \ldots \ldots n$) are α-admissible contractions on complete metric space (X, d).

Lemma 1: Let$\{f_1, f_2, f_3 \ldots \ldots \ldots f_n\}$ be an IFS on complete metric space (X, d). The mapping $\mathcal{F}: K(X) \to K(X)$ is α-admissible if the system of functions $\{f_1, f_2, f_3 \ldots \ldots \ldots f_n\}$ is α-admissible.

Proof: Since $\{f_1, f_2, f_3 \ldots \ldots \ldots f_n\}$ are α-admissible so $\alpha(x, y) \geq 1 \Rightarrow \alpha(f_i(x), f_i(y)) \geq 1$

For $i = 1,2,3 \ldots \ldots n$

Now by definition (3.1.2) $\alpha(x, y) \geq 1 \;\Rightarrow \alpha(A, B) \geq 1$for all $x \in A$, $y \in B$ and $A, B \in K$.

Since $f_1, f_2, f_3 \ldots \ldots \ldots f_n$ are α-admissible thus $\alpha(x, y) \geq 1 \Rightarrow \alpha\big(f_i(x), f_i(y)\big) \geq 1 \; \forall \; f_i(x) \in \mathcal{F}(A) and \; f_i(y) \in \mathcal{F}(B)$, this implies that $min.\,\alpha\big(f_i(x), f_i(y)\big) \geq 1 \;\Rightarrow\; \alpha(F(A), F(B)) \geq 1$. Thus $\alpha(A, B) \geq 1 \Rightarrow \alpha\big(\mathcal{F}(A), \mathcal{F}(B)\big) \geq 1$ .Hence $\mathcal{F}$ is α-admissible in (K, H_d).

$\mathcal{F}$ Satisfies $\alpha - \varphi$ contraction

Definition 9: (Rus et al., 2008)The real valued function $\varphi: [0, \infty) \to [0, \infty)$ which satisfies following properties;

a. φ is monotonically non-decreasing ,right continuous and $\varphi(t) < t \; \forall \; t > 0$
b. The infinite iterates $\lim_{n \to \infty} \varphi^n(t) = 0$ for any $t > 0$

Such function is called comparison function and the set of such functions is denoted by $\emptyset$

Definition 10: Let f be a self- mapping in a complete metric space (X, d) then f is called $\alpha - \varphi$ contraction if there exist two functions $\alpha: X \times X \to [0, \infty)$ and $\varphi \in \emptyset$ such that

$\alpha(x, y) d(fx, fy) \leq \varphi(dx, y)) \; \forall \; x, y \in X$

Proposition 1: If for each $a_i \in A \; \exists b_i \in B$ such that $d(a_i, b_i) < \epsilon \Rightarrow H_d(A, B) < \epsilon$

Proof: If for each $a_i \in A \; \exists b_i \in B$ such that $d(a_i, b_i) < \epsilon$ then by symmetry of metric d for each $b_i \in B \; \exists a_i \in A$ such that $d(a_i, b_i) < \epsilon$ thus by definition of H_d we must have $H_d(A, B) < \epsilon$.

Now we show $\mathcal{F}$ is $\alpha - \varphi$ contraction;

Lemma 2: Let (X, d) is complete metric space. Let $\{(X, d); f_1, f_2, f_3 \ldots \ldots \ldots f_n\}$ be an IFS with each $f_i \; is \; \alpha - \varphi_i$ contraction. Then the operator $\mathcal{F}(A) = \bigcup_{i=1}^{n} f_i(A)$ is also $\alpha - \varphi$ contraction. Where, $\varphi = max.\{\varphi_i \in \emptyset\}$.

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Proof: Here, $\varphi = max.\{\varphi_i \in \emptyset\}$, that is $\varphi(t) = max.\{\varphi_1(t), \varphi_1(t), \varphi_1(t) \dots \dots \varphi_n(t)\} < max.(t, t, t \dots . t\} = t$.Thus φ is right continuous and increasing. For any $A \in K(X)$ is compact subset of X then $\mathcal{F}(A)$ is also compact subset of X because each member f_i is continuous. Thus, $\mathcal{F}: K(X) \to K(X)$ is a self-map. Now take any $A, B \in K(X)$ and choose $\epsilon > 0$ such that; $H_d(A, B) < \epsilon$ .Then for each $a_j \in A \exists b_j \in B$ such that $d(a_j, b_j) < \epsilon$. Since each member f_i is $\alpha - \varphi_i$ contraction thus we have;

$$\alpha(a_j, b_j)d(f_i(a_j), f_i(b_j)) \leq \varphi_i(d(a_j, b_j)) < \varphi_i(\epsilon) < \varphi(\epsilon)$$

Since $\alpha(A, B) = \min\{ \alpha(a_j, b_j)\}$ thus;

$\alpha(A, B)d(f_i(a_j), f_i(b_j) < \varphi(\epsilon)$. This is true for each $a_j \in A \exists b_j \in B$, thus by proposition (3.1.1) We must have; $\alpha(A, B)H_d(\mathcal{F}(A), \mathcal{F}(B)) < \varphi(\epsilon)$. Now taking $\epsilon \to H_d(A, B)$ and as φ is right continuous, then we have;

$$\alpha(A, B)H_d(\mathcal{F}(A), \mathcal{F}(B)) < \varphi(H_d(A, B)) \quad \dots\dots\dots\dots (1)$$

Hence $\mathcal{F}$ is $\alpha - \varphi$ contraction.

Fixed point theorem

Before to prove the main theorem, we re- state the theorem of B. Samet et al.(Samet et al., 2012)

Theorem 1: Let (X, d) be a complete metric space and $f: X \to X$ be a self- mapping. Suppose that there exist $\alpha: X \times X \to [0, \infty)$ and $\varphi \in \emptyset$ such that;

(i) f is $\alpha - \varphi$ contraction
(ii) f is α-admissible
(iii) There exists $x_0 \in X$ such that $\alpha(x_0, f^m x_0) \geq 1 \forall m \in N$
(iv) If for every sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$for $n \in N$ then $\alpha(x_n, x) \geq 1 \ \forall n \in N$
(v) If for all $x, y \in X \exists u \in X$ such that $\alpha(x, u) \geq 1 \ and \ \alpha(y, u) \geq 1$

Then f has unique fixed point in X.

Now we prove following main theorem;

Theorem 2: Let (X, d) is complete metric space. Let $\{(X, d); f_1, f_2, f_3 \dots \dots \dots f_n\}$ be an IFS with each $f_i \ is \ \alpha - \varphi_i$ contraction and α-admissible. Then the operator

$\mathcal{F}(A) = \bigcup_{i=1}^{n} f_i(A) \ \forall A \in K(X)$ Satisfies following conditions;

(i) By lemma (2), $\mathcal{F} \ is \ \alpha - \varphi$ Contraction. Where, $\varphi = max.\{\varphi_i \in \emptyset\}$.
(ii) By lemma (1), $\mathcal{F}$ is α-admissible
(iii) Suppose there exist $A_0 \in K(X)$ such that $\alpha(A_0, \mathcal{F}^m(A_0)) \geq 1, m \in N$
(iv) Suppose for every sequence $\{A_n\} \subset K(X)$ such that $A_n \to A \in K(X)$ and if $\alpha(x_n, x_{n+1}) \geq 1 \ \forall x_n \in A_n$ then $\alpha(A_n, A_{n+1}) \geq 1$ and $\alpha(A_n, A) \geq 1 \ \forall n \in N$.

(v) Suppose for all $A, B \in K(X)$ there exist $C \in K(X)$ such that $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1\ \forall\ x \in A, y \in B\ and\ z \in C$ then $\alpha(A, C) \geq 1$ and$\alpha(B, C) \geq 1$, then $\mathcal{F}$ has unique fixed point in $K(X)$.

Proof: Since (X, d) is complete metric space so by Hutchison (Hutchinson, 1981) the space $(K(X), H_d)$ is also complete metric space and any IFS $\{(X, d); f_1, f_2, f_3 \dots \dots \dots f_n\}$ generates a self-map $\mathcal{F}: K(X) \to K(X)$, by lemma 2, $\mathcal{F}\ is\ \alpha - \varphi$ Contraction if each $f_i\ is\ \alpha - \varphi_i$ contraction. Let $A_0 \in K(X)$ such that $\alpha\big(A_0, f^m(A_0)\big) \geq 1, \forall\ m \in N$.Let us define sequence $\{A_n\}$ by $A_1 = \mathcal{F}(A_0)$ and $A_{n+1} = \mathcal{F}(A_n) \forall\ n \in N$. If $A_{n+1} = A_n$ for some $n \in N$ then A_n is fixed point of $\mathcal{F}$.So let$A_{n+1} \neq A_n\ \forall n \in N$. Since $\mathcal{F}$ is α-admissible so from (iii)

$\alpha\big(A_0, \mathcal{F}(A_0)\big) \geq 1$

$\Rightarrow \alpha(\mathcal{F}(A_0), \mathcal{F}^2(A_0)) \geq 1$ [$\because F$ is α-admissible]

$\Rightarrow \alpha(\mathcal{F}^2(A_0), \mathcal{F}^3(A_0)) \geq 1$

…………………………..

$\Rightarrow \alpha(\mathcal{F}^n(A_0), \mathcal{F}^{n+1}(A_0)) \geq 1$

$\Rightarrow \alpha(A_n, A_{n+1}) \geq 1\ \forall n \in N$

Now for any $n \in N$

$H_d(A_n, A_{n+1}) = H_d(\mathcal{F}(A_{n-1}), \mathcal{F}(A_n))$

$\leq \alpha(A_{n-1}, A_n) H_d(\mathcal{F}(A_{n-1}), \mathcal{F}(A_n))$ [$\because\ \alpha(A_{n-1}, A_n) \geq 1$

$\leq \varphi(H_d(\mathcal{F}(A_{n-1}), \mathcal{F}(A_n)))$ [$\because F\ is\ \alpha - \varphi$ Contraction]

$\leq \varphi^2(H_d(\mathcal{F}(A_{n-2}), \mathcal{F}(A_{n-1})))$

……………………………………..

$\leq \varphi^n(H_d(\mathcal{F}(A_0), \mathcal{F}(A_1)))$

Taking $n \to \infty$ then by properties of φ,

$\varphi^n(H_d(\mathcal{F}(A_0), \mathcal{F}(A_1))) = 0$ [$\because H_d(\mathcal{F}(A_0), \mathcal{F}(A_1)) > 0$]

Thus $H_d(A_n, A_{n+1}) = 0\ \ \forall n \in N$……………………… (*)

Again for any $m, n \in N$ $(n > m)$

$$H_d(A_m, A_n) \leq H_d(A_m, A_{m+1}) + H_d(A_{m+1}, A_{m+2}) + \cdots \dots \dots \dots + H_d(A_{n-1}, A_n)$$

Taking $m \to \infty$ then $n \to \infty$ and from (*)

$H_d(A_m, A_n) = 0$.thus $\{A_n\}$is Cauchy sequence. Since $(K(X), H_d)$ complete, so there exist $A \in K(X)$ such that $A_n \to A$ i.e. $\lim_{n\to\infty} H_d(A_n, A) = 0$……………….. (**)

From condition (iv) $\alpha(A_n, A_{n+1}) \geq 1 \Rightarrow\ \alpha(A_n, A) \geq 1\ \forall n \in N$. Now,

$$H_d(A_{n+1}, \mathcal{F}(A)) = H_d(\mathcal{F}(A_n), \mathcal{F}(A))$$

$$\leq \alpha(A_n, A) H_d(\mathcal{F}(A_n), \mathcal{F}(A))$$

$$\leq \varphi(H_d(A_n, A)) \to 0 \text{ , as } n \to \infty \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ (***)}$$

Again, $H_d(A, \mathcal{F}(A)) \leq H_d(A, A_n) + H_d(A_n, A_{n+1}) + H_d(A_{n+1}, \mathcal{F}(A))$

Taking $n \to \infty$ and from (*),(**) and (***) we have, $H_d(A, \mathcal{F}(A)) = 0$. Thus $\mathcal{F}(A) = A$

To show uniqueness, let B is another fixed point so $\mathcal{F}(B) = B$.Now by condition (v), for $A, B \in K(X)$ $\exists$ $C \in K(X)$ such that $\alpha(A, C) \geq 1$ and $\alpha(B, C) \geq 1$.Since $\mathcal{F}$ is α-admissible, so

$\alpha(A, C) \geq 1 \Rightarrow \alpha(A, \mathcal{F}^n(C)) \geq 1$.

Now,

$$H_d\big(A, \mathcal{F}^n(C)\big) = H_d(\mathcal{F}(A), \mathcal{F}(\mathcal{F}^{n-1}(C)))$$

$$\leq \alpha(A, \mathcal{F}^{n-1}(C)) H_d(\mathcal{F}(A), \mathcal{F}(\mathcal{F}^{n-1}(C)))$$

$$\leq \varphi(H_d\big(A, \mathcal{F}^{n-1}(C)\big) \leq \varphi^2(H_d\big(A, \mathcal{F}^{n-2}(C)\big) \ldots\ldots \leq \varphi^n(H_d(A, \mathrm{C}))$$

Taking $n \to \infty$,$\varphi^n(H_d(A, \mathrm{C})) \to 0$ .Therefore $H_d\big(A, \mathcal{F}^n(C)\big) = 0$

Similarly, $H_d\big(B, \mathcal{F}^n(C)\big) = 0$ as $n \to \infty$

Now, $H_d(A, B) \leq H_d\big(A, \mathcal{F}^n(C)\big) + H_d(\mathcal{F}^n(C), B)$, When $n \to \infty$ then $H_d(A, B) = 0$

Therefore $A = B$.Hence fixed point is unique.

Conclusion

We noticed following results from references herein;

(a) If (X, d) is complete metric space then $(K(X), H_d)$ is also complete metric space.
(b) If (X, d) is compact metric space then $(K(X), H_d)$ is also compact metric space.
(c) If there is finite set of contractions $f_1, f_2, f_3 \ldots\ldots\ldots f_n$ in (X, d) then $\mathcal{F}: K(X) \to K(X)$ induced by $f_1, f_2, f_3 \ldots\ldots\ldots f_n$ is also contraction in $(K(X), H_d)$ with contraction factor maximum of contraction factor of $f_1, f_2, f_3 \ldots\ldots\ldots f_n$.
(d) If the self-mapping set $f_1, f_2, f_3 \ldots\ldots\ldots f_n$ in (X, d) is φ_i-contractions then the mapping $\mathcal{F}: K(X) \to K(X)$ is also φ-contractionin$(K(X), H_d)$ where φ is max.$\{\varphi_i\}$ for $i = 1,2, \ldots\ldots n$.
(e) If (X, d) is semi-metric space then $(K(X), H_d)$ is also semi-metric space.

By view of above conditions we found that "If the self-mapping set $f_1, f_2, f_3 \ldots\ldots\ldots f_n$ in (X, d) is $\alpha - \varphi_i$-contractions then the mapping $\mathcal{F}: K(X) \to K(X)$ is also $\alpha - \varphi$-contraction, where φ is max.$\{\varphi_i\}$ .For $i = 1,2, \ldots\ldots n$ and there exist a unique fixed point of $\mathcal{F}$ in $(K(X), H_d)$."

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