Review of Some Identities Involving Basic Hypergeometric Series

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ABSTRACT

In this paper, we review and verify several identities involving basic hypergeometric series including the q- binomial theorem. These identities play a fundamental role in the theory of q-series and may serve as a useful reference for beginners interested in this area of mathematics.

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1. INTRODUCTION, NOTATION AND DEFINITIONS

In 1812, Gauss [1-5] presented to the Royal Society of Sciences at Göttingen his famous paper, in which he considered the infinite series

$$1+\frac{a.b}{1.c}z+\frac{a(a+1)b(b+1)}{1.2.c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}z^3+\ldots \quad (1.1)$$

as function of *a, b, c, z* where it is assumed that $c \neq 0,-1,-2,-3,\ldots$ .He showed that the series converges absolutely for $|z|<1$ and for $|z|$=1when Re (c-a-b) > 0. This series is denoted by ${}_2F_1\left[\begin{matrix}a,b;z\\c\end{matrix}\right]$.

$$Thus\ {}_2F_1\left[\begin{matrix}a,b;z\\c\end{matrix}\right]=1+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n n!} for\ c \neq 0,-1,-2,-3,\ldots. \quad (1.2)$$

$$where\ (a)_n=a(a+1)(a+2)\ldots(a+n-1), n \geq 1\ and\ (a)_0=1$$

$If\ a=1, b=c,$ series (1.1) yields thegeometric series

$$1+z+z^2+z^3+\ldots, |z|<1 \quad (1.3)$$

$If\ b=c,$ series (1.1) reduces to binomial theorem

$$1+\frac{a}{1!}z+\frac{a(a+1)}{2!}z^2+\ldots=(1-z)^{-a}, |z|<1. \quad (1.4)$$

Another generalization of Gauss series [2-7] is the generalized hypergeometric series with r numerator parameters $a_1,a_2,\ldots,a_r$ and s denominator parameters $b_1,b_2,\ldots,b_s$ defined by

$${}_rF_s\left[\begin{matrix}a_1,a_2,\ldots,a_r;z\\b_1,b_2,\ldots,b_s\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_r)_n z^n}{n!\,(b_1)_n(b_2)_n\ldots(b_s)_n} \quad (1.5)$$

in which no denominator parameters $b_1,b_2,\ldots,b_s$ are allowed to be zero or negative integers. Many other mathematicians studied similar series, notably the Swiss L. Euler [8], A. T. Vandermonde [9] Slater [6, 7]. After thirty-three years of Gauss' paper, Heine [10-12] introduced the series

$$1+\frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}z+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}z^2+\ldots \quad (1.6)$$

$$where\ c \neq 0,-1,-2,\ldots$$

Series (1.6) converges absolutely for $|z|<1$ when $|q|<1$. Series (1.6) tends to Gauss' series(1.1) as $q \to 1$, due to $\lim_{q\to 1}\frac{1-q^n}{1-q}$

The series (1.6) is called Heine's series or basic hypergeometric series or q-hypergeometric series and is denoted by ${}_2\phi_1\left[\begin{matrix}a,b;q,z\\c\end{matrix}\right]$.

$$Thus,\ {}_2\phi_1\left[\begin{matrix}a,b;q,z\\c\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n \quad (1.7)$$

$$where\ (a;q)_n=\begin{cases}1, & n=0\\(1-a)(1-aq)\ldots(1-aq^{n-1}), & n=1,2,3,\ldots\end{cases} \quad (1.8)$$

$is\ q-$ shifted factorial.

For brevity,
let $(a_1,a_2,\ldots,a_r;q)_n=(a_1;q)_n(a_2;q)_n\ldots(a_r;q)_n$

The generalized basic hypergeometric [13] series is defined by

$$ {}_r\phi_s \begin{bmatrix} a_1, a_2, \dots a_r; q, z \\ b_1, b_2, \dots b_s \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n z^n}{(b_1, b_2, \dots, b_s; q)_n} \quad (1.9) $$

$$ where \ \binom{n}{2} = \frac{n(n-1)}{2} $$

The infinite series in (1.9) is absolutely convergent for all values of $z \ if \ r \le s \ and \ for \ r = s+1$, it converges in the region $|z| < 1$.

1.1 Identities frequently used in basic hypergeometric series

In this section we shall discuss about the following identities (1.2.30 - 1.2.40), p.6) [3]. These identities are frequently used in number theory [13] such as partition and mock theta functions. The Notebooks of Ramanujan and his 'Lost' Notebook, containing about 4000 Entries/theorem will continue to be eternal sources of inspiration to the mathematicians of the world. Ramanujan did not provide proofs of maximum such Entries/ theorem. Andrews [14-18], Slater [6, 7] and many other mathematicians gave the proofs of some such Entries/theorem. Anyone who interested in these branches of mathematics, the identities mentioned below are crucial prerequisites for studying such matter. To prove these identities q-factorial relation mentioned in (1.8) is used.

(a) $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$ ... (2.1)

Proof: $(a;q)_n = (1-a)(1-aq)(1-aq^2) \dots (1-aq^{n-1})$

$$ = \frac{(1-a)(1-aq)(1-aq^2) \dots (1-aq^{n-1})(1-aq^n)(1-aq^{n+1})}{(1-aq^n)(1-aq^{n+1}) \dots \ to\infty} $$

$$ = \frac{(a;q)_\infty}{(aq^n;q)_\infty} $$

(b) $(a^{-1}q^{1-n};q)_n = (a;q)_n(-a^{-1})^n q^{-\binom{n}{2}}$ ... (2.2)

Pr o of:

$$ (a^{-1}q^{1-n};q)_n = (1-a^{-1}q^{1-n})(1-a^{-1}q^{1-n}q)(1-a^{-1}q^{1-n}q^2)\dots(1-a^{-1}q^{1-n}q^{n-1}) $$

$$ = (1-a^{-1}q^{1-n})(1-a^{-1}q^{2-n})(1-a^{-1}q^{3-n})\dots(1-a^{-1}q^{-1})(1-a^{-1}) $$

$$ = \frac{(-1)^n(1-a)(1-aq)\dots(1-aq^{n-1})}{a^n q^{\frac{n(n-1)}{2}}} $$

$$ = (a;q)_n(-a^{-1})^n q^{-\binom{n}{2}} $$

(c) $(a;q)_{n-k} = \frac{(a;q)_n(-qa^{-1})^k q^{\binom{k}{2}-nk}}{(a^{-1}q^{1-n};q)_k}$ ... (2.3)

Proof: $(a;q)_{n-k} = (1-a)(1-aq)(1-aq^2) \dots (1-aq^{n-k-2})(1-aq^{n-k-1})$

$$ = \frac{(1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-k-1})(1-aq^{n-k})..(1-aq^{n-1})}{(1-aq^{n-k})(1-aq^{n-k+1})\dots(1-aq^{n-1})} $$

$$ = \frac{(a;q)_n}{\frac{(-1)^k(1-a^{-1}q^{1-n})(1-a^{-1}q^{2-n})\dots(1-a^{-1}q^{k-n})}{(a^{-1})^k q^{\frac{k(k+1)}{2}-nk}}} $$

$$ = \frac{(a;q)_n(-qa^{-1})^k q^{\binom{k}{2}-nk}}{(a^{-1}q^{1-n};q)_k}. $$

(d) $(a;q)_{n+k} = (a;q)_n(aq^n;q)_k$ (2.4)

Proof:

$$ (a;q)_{n+k} = (1-a)(1-aq)(1-aq^2)\dots(1-aq^n)\dots(1-aq^{n+k-1}) = (a;q)_n(aq^n;q)_k $$

(e) $(aq^n;q)_k = \frac{(a;q)_k(aq^k;q)n}{(a;q)n}$ (2.5)

Proof: $(aq^n;q)_k = (1-aq^n)(1-aq^nq)\dots(1-aq^nq^{k-1})$

$$ = \frac{(1-a)(1-aq)\dots(1-aq^{n-1})(1-aq^n)(1-aq^nq)\dots(1-aq^nq^{k-1})}{(1-a)(1-aq)\dots(1-aq^{n-1})} $$

$$ = \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n} $$

(f) $(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}$ (2.6)

Proof: $(aq^k;q)_{n-k}$

$$ = (1-aq^k)(1-aq^kq)\dots(1-aq^kq^{n-k-1}) $$

$$ = \frac{(1-a)(1-aq)\dots(1-aq^{k-1})(1-aq^k)(1-aq^{k+1})\dots(1-aq^{k+n-k-1})}{(1-a)(1-aq)\dots(1-aq^{k-1})} $$

$$ = \frac{(a;q)_n}{(a;q)_k} $$

(g) $(aq^{2k};q)_{n-k} = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}$ (2.7)

Proof: $(aq^{2k};q)_{n-k}$

$$ = (1-aq^{2k})(1-aq^{2k}q)\dots(1-aq^{2k+n-k-1}) $$

$$ = \frac{(a;q)_{2k}(1-aq^{2k})(1-aq^{2k}q)\dots(1-aq^nq^{k-1})}{(a;q)_{2k}} $$

$$ = \frac{(1-a)(1-aq)\dots(1-aq^{n-1})(1-aq^n)\dots(1-aq^nq^{k-1})}{(a;q)_{2k}} $$

$$ = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}} $$

(h) $(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}$ (2.8)

Proof: $(q^{-n};q)_k = (1-q^{-n})(1-q^{-n}q)\dots(1-q^{-n}q^{k-1})$

$$ = \frac{(-1)^k(1-q^n)(1-q^{n-1})\dots(1-q^{n-k+1})}{q^{\frac{k(k-1)}{2}-nk}} $$

$$ = \frac{(-1)^k(q;q)_{n-k}(1-q^{n-k+1})(1-q^{n-k+2})\dots(1-qq^{n-1})}{q^{\frac{k(k-1)}{2}-nk}(q;q)_{n-k}} $$

$$ = \frac{(q;q)_n(-1)^k q^{\binom{k}{2}-nk}}{(q;q)_{n-k}} $$

(i) $(aq^{-n};q)_k = \frac{(a;q)_k(qa^{-1};q)_n q^{-nk}}{(a^{-1}q^{1-k};q)_n}$ (2.9)

Proof: $(aq^{-n};q)_k = \frac{(aq^{-n};q)_\infty}{(aq^{k-n};q)_\infty}$

$$ = \frac{(a;q)_\infty}{(a;q)_{-n}}\frac{(a;q)_{k-n}}{(a;q)_\infty} = \frac{(a;q)_{k-n}}{(a;q)_{-n}} $$

$$= \frac{\left(\frac{q}{a};q\right)_n (a;q)_k (aq^k;q)_{-n}}{(-1)^n a^{-n} q^{\frac{n(n+1)}{2}}}$$

$$= \frac{(qa^{-1};q)_n (a;q)_k}{(-1)^n a^{-n} q^{\frac{n(n+1)}{2}}} \times \frac{(-1)^n (aq^k)^{-n} q^{\frac{n(n+1)}{2}}}{\left(\frac{q}{aq^k};q\right)_n}$$

$$= \frac{(a;q)_k (qa^{-1};q)_n q^{-nk}}{(a^{-1}q^{1-k};q)_n}$$

(j) $(a;q)_{2n} = (a;q^2)_n (aq;q^2)_n$ ... (2.10)

Proof: $(a;q)_{2n} = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{2n-1})$

$= (1-a)(1-aq^2)(1-aq^4)\dots(1-aq^{2n-2}) \times (1 - aq)(1-aq^3)(1-aq^5)\dots(1-aq^{2n-1}$

$= (a;q^2)_n (aq;q^2)_n$

(k) $(a^2;q^2)_n = (a;q)_n(-a;q)_n$ ... (2.11)

Proof: $(a^2;q^2)_n = (1-a^2)(1-a^2q^2)\dots(1-a^2q^{2(n-1)})$

$= \{(1-a)(1-aq)\dots(1-aq^{n-1})\} \times \{(1+a)(1+aq)\dots(1-aq^{n-1})\}$

$= (a;q)_n(-a;q)_n$

1.2 The q- binomial theorem

The most fundamental summation formula in the theory of basic hypergeometric series is the q- binomial theorem [3 13]

$${}_1\phi_0[a;-;q,z] = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, |z|<1, |q| < 1 \quad (3.1)$$

Proof: *Let* $f(z) = \frac{(az;q)_\infty}{(z;q)_\infty}$ ... (3.2)

Since $f(z)$ is an analytic function in $|z|<1$, so we have its taylor's expansion in the form

$$f(z) = \sum_{n=0}^{\infty} A_n z^n \quad \dots (3.3)$$

$$f(zq) = \frac{(azq;q)_\infty}{(zq;q)_\infty} = \frac{(1-z)}{(1-az)} \frac{(az;q)_\infty}{(z;q)_\infty}$$

$\Rightarrow (1-az)f(zq) = (1-z)f(z)$

$$\Rightarrow (1-az)\sum_{n=0}^{\infty} A_n z^n q^n = (1-z)\sum_{n=0}^{\infty} A_n z^n \quad (3.4)$$

Equating the coefficients of z^n on both, we get

$A_n q^n - aA_{n-1}q^{n-1} = A_n - A_{n-1}$

$\Rightarrow A_n(1-q^n) = A_{n-1}(1-aq^{n-1})$

$$\Rightarrow A_n = \left(\frac{1-aq^{n-1}}{1-q^n}\right) A_{n-1} \quad (3.5)$$

From (3.5) *we find*

$$A_1 = \frac{1-a}{1-q}A_0, A_2 = \frac{1-aq}{1-q^2}A_1 = \frac{(1-a)(1-aq)}{(1-q)(1-q^2)}A_0$$

Proceeding in this way, we get

$$A_n = \frac{(1-a)(1-aq)\dots(1-aq^{n-1})}{(1-q)(1-q^2)\dots(1-q^n)}A_0 = \frac{(a;q)_n}{(q;q)_n}A_0 \quad (3.6)$$

From (3.3), $f(0) = A_0$ and from (3.2) $f(0) = 1$, *so* $A_0 = 1$

Putting this value in (3.6), we get $A_n = \frac{(a;q)_n}{(q;q)_n}$

Using this value in (3.3), we get

$$f(z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}$$

1.3 Some Special unsolved problem [13] obtained by using section (1) and (2)

(i) $(aq^{-n};q)_n = \left(\frac{q}{a};q\right)_n \left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}$

Proof: $(aq^{-n};q)_n$

$= (1-aq^{-n})(1-aq^{-n}q)\dots(1 - aq^{-n}q^{n-2})(1-aq^{-n}q^{n-1})$

$$= \left(\frac{q}{a}-1\right)\left(\frac{q^2}{a}-1\right)\dots\left(\frac{q^{n-1}}{a}-1\right)\left(\frac{q^n}{a}-1\right)\frac{a^n}{q^{\frac{n(n+1)}{2}}}$$

$$= \left(\frac{q}{a};q\right)_n \left(-\frac{a}{q}\right)^n q^{-\binom{n}{2}}.$$

(ii) $(aq^{-k-n};q)_n = \frac{(q/a;q)_{n+k}(-a)^n q^{\binom{n}{2} - nk}}{(q/a;q)_k}$

Proof: $(aq^{-k-n};q)_n = (1-aq^{-k-n})(1-aq^{-k-n+1})\dots\times (1-aq^{-k-n+n-2})(1-aq^{-k-n+n-1})$

$$= \frac{(-a)^n\left(1-\frac{q^{k+1}}{a}\right)\left(1-\frac{q^{k+2}}{a}\right)\dots\left(1-\frac{q^{k+n-1}}{a}\right)\left(1-\frac{q^{k+n}}{a}\right)}{q^{kn+\frac{n(n-1)}{2}}}$$

$$= \frac{(q/a;q)_{n+k}}{(q/a;q)_k}\frac{(-a)^n q^{-nk}}{q^{n(n-1)/2}} = \frac{(q/a;q)_{n+k}(-a)^n q^{\binom{n}{2}-nk}}{(q/a;q)_k}$$

(iii) $\frac{(qa^{1/2},-qa^{1/2};q)_n}{(a^{1/2},-a^{1/2};q)_n} = \frac{1-aq^{2n}}{1-a}$

Proof: $\frac{(qa^{1/2},-qa^{1/2};q)_n}{(a^{1/2},-a^{1/2};q)_n}$

$$= \frac{(1-qa^{1/2})(1-q^2a^{1/2})\dots(1-q^na^{1/2})(1+qa^{1/2})(1+q^2a^{1/2})\dots(1+q^na^{1/2})}{(1-a^{1/2})(1-qa^{1/2})\dots(1-q^{n-1}a^{1/2})(1+a^{1/2})(1+qa^{1/2})\dots(1+q^{n-1}a^{1/2})}$$

$$= \frac{(1-q^2a)(1-q^4a)\dots(1-q^{2(n-1)}a)(1-q^{2n}a)}{(1-a)(1-q^2a)\dots(1-q^{2(n-1)}a)}$$

$$= \frac{1-aq^{2n}}{1-a}$$

(iv) $(a;q)_\infty = (a^{1/2},-a^{1/2},(aq)^{1/2},-(aq)^{1/2})_\infty$

Proof: $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)(1-aq^3)(1 - aq^4)\dots$ *to* ∞

$= (1-(a^{1/2})^2)(1-\{(aq)^{1/2}\}^2)(1-\{(a^{1/2}q)\}^2)\dots$ *to* ∞

$= \{(1-a^{1/2})(1-a^{1/2}q)\dots$ *to*$\infty\}\{(1+a^{1/2})(1 + a^{1/2}q)\dots$ *to*$\infty\} \times$

$\{(1-(aq)^{1/2})(1-(aq)^{1/2}q)\dots$ *to*$\infty\}\{(1+(aq)^{1/2})(1 + (aq)^{1/2}q)\dots$ *to*$\infty\}$

$= (a^{1/2} - a^{1/2}(aq)^{1/2} - (aq)^{1/2})_\infty$

2. CONCLUSION

In the paper, we reviewed some identities of basic hypergeometric series which are essential for entering the higher q- series. Without those identities we can't enter to study the Ramanujan's work. Those identities and q-binomial theorem are proved by using the q-shifted factorial mentioned in (1.8). These identities are necessary but not sufficient for such higher study. Also, in the paper, with the help of those identities, some unsolved problem of Gasper and Rahman are solved. We hope this paper will help to attract the beginners in this branch of mathematics.

AUTHOR CONTRIBUTION

Whole work was carried by GS Pant.

CONFLIT OF INTEREST

Authors declare no competing interests with respect to the publication of this manuscript.

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