On Some Properties of A_p Weights

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ABSTRACT

We provide a survey of the literature on the issue of weighted inequalities of operators in harmonic analysis. Some features of A_p weight functions are discussed and elaborated in this note that will assist in the comprehensive study on the theory of weights.

1. NTRODUCTION

1.1 Background and Literature Review

The concept of A_p weight was introduced by Muckenhoupt [1] in 1972. Weighted inequalities deals with boundedness features of an operator T on weighted Lebesgue space $L^p(w)$, where T is commonly maximal function, square function and other classical operators of harmonic analysis. Basically strong and weak-type estimates of the form

$\int_{\mathbb{R}^n}|Tf(x)|^p\, u(x)dx \leq c\int_{\mathbb{R}^n}|f(x)|^p\, v(x)dx$ and

$u(\{x\in\ \mathbb{R}^n: |Tf(x)|>\lambda\}) \leq \frac{c}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p\, v(x)dx$ are

included in the study of weighted norm inequalities. Either a single weight $(u=v=w)$ or a pair of weights (u,v) could be the issue [2].

Muckenhoupt [1] first proved that both the weighted strong and weak type estimates for Hardy-Littlewood maximal function M holds if and only if w satisfies the A_p condition. The same class of weights also characterizes the boundedness of the Hilbert transform on $L^p(w)$, was proved by Hunt, Muckenhoupt and Wheeden [4] shortly after. In 1974, Coifman and Fefferman [5] proved that the general Calderón-Zygmund operators and their analogues are bounded on $L^p(w)$ whenever w satisfies the A_p condition. In addition, the proof of the boundedness of maximal operators and singular integral operators on weighted spaces were built on the foundation of A_∞ condition, the reverse Hölder's inequality and A_p implies $A_{p-\varepsilon}$ that are based on A_p condition.

Discussing and elaborating on certain aspects of A_p weight functions that could improve knowledge of weight theory is the primary goal of this work.

Analytical methods and basic analysis tools are used to present the work on the spirit of close to the books [9, 10].

1.2 Weighted Norm Inequalities

For the past three decades, mathematicians have studied the connection between the so-called A_p characteristic of the weight and the norm of certain operators for singular integrals in a weighted space in the strong type (p,p) inequality.

The Jones factorization theorem and the Rubio de Francia Extrapolation theorem are two essential and closely connected findings in the study of weighted norm inequalities. Jones factorization theorem resembles with the property 3.4.

The study of optimal quantitative estimates for the norm, $\|T\|_{L^p(w)}$, whenever $w\in A_p$ is a crucial topic in theory of weighted norm inequalities.

Buckley [6] first studied the question for the Hardy-Littlewood maximal operator M, and proved the estimate

$\|Mf\|_{L^p(w)} \leq\ C_{n,p}[w]_{A_p}^{1/p-1}\|f\|_{L^p(w)}.$

For the chronology of the linear estimate of the type $\|Tf\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}$ we refer, page 173 in [7].

1.3 Significance of the Study

Weighted estimates holds significant importance in several domains, including Fourier analysis, complex analysis, approximation theory, partial differential equations, theory of quasi-conformal mappings, operator theory, regularity theory of Beltrami equations [7]. The idea of $A_p$weight is fundamental to weight theory and serves as a stepping stone. For more detail of weight theory, we refer [8, 9, 10, 11, 12].

2 . BASIC NOTATIONS AND DEFINITIONS

The necessary notations and definitions used in this work are as follows.

2.1 Notations

- $\mathrm{w(E)} = \int_{\mathrm{E}}\mathrm{w(x)\,dx}$ denotes the w-measure of the set E
- $|Q|$ is Lebesgue measure of the set Q
- $\langle w\rangle_Q = \frac{1}{|Q|}\int_Q w(x)dx$
- $L^p(\mathbb{R}^n, w)$ or simply $L^p(w) := \{f: \|f\|_{L^p(\mathbb{R}^n,w)} < \infty$ denotes the weighted Lp space, where $\|f\|_{L^p(\mathbb{R}^n,w)} := \left(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,dx\right)^{\frac{1}{p}}.$

2.2 Definitions

Definition 2.2.1

A weight is locally integrable function on $\mathbb{R}^n$ that takes values

in the interval $(0, \infty)$ almost everywhere. So by definition a weight function can be zero or infinity only on a set whose Lebesgue measure is zero and w-measure of the measurable set E is defined as
$w(E) = \int_E w(x)\,dx$ [9, 10].

Definition 2.2.2
Let, $1 \le p < \infty$. A weight, $w \in L^1_{loc}(\mathbb{R}^n)$, with $w(x) \ge 0$ for almost every $x \in \mathbb{R}^n$ is said to be an A_p weight if there exists a constant C, independent of Q, such that
$\frac{1}{|Q|}\int_Q w(x)dx \left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right) \le C$ for every $Q \in \mathbb{R}^n$.
The Muckenhoupt characteristic constant or simply A_p constant of w represented by $[w]_{A_p}$ is the smallest constant C for which the inequality is true, where the supremum is taken over all the cubes whose sides are parallel to coordinate axes. Thus
$$[w]_{A_p} := Sup_{Q in\, \mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1}.$$
w is said to be in A_p class and we denote $w \in A_p$ [7, 9, 10, 11, 12, 13].

Definition 2.2.3
A finite, strictly positive and locally integrable function w is said to be an A_1 weight or in A_1 class if there exists a constant $C > 0$ such that $Mw(x) = Cw(x)$ for $x \in \mathbb{R}^n$ a.e. and M is uncentered Hardy Littlewood maximal function. $[w]_{A_1}$ is the minimum of C satisfying the inequality [9, 10, 13].

Definition 2.2.4
Given a locally integrable function f on $\mathbb{R}^n$ the Hardy-Littlewood maximal function Mf of f is defined by $Mf(x) := Sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,dy, x \in \mathbb{R}^n$, where the supremum is taken over all cubes containing x. The operator, $M: f \to Mf$ is called Hardy-Littlewood maximal operator [8, 10, 11].

3. PROPERTIES [9, 10]

Property 3.1
$[\tau^z(w)]_{A_p} = [w]_{A_p}$, where $\tau^z(w)(x) = w(x - z), z \in \mathbb{R}^n$.
Proof: We have by definition,
$$[\tau^z(w)]_{A_p} = Sup_{Q \text{ in } \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q \tau^z(w)\,dx\right)\left(\frac{1}{|Q|}\int_Q \tau^z(w)^{-\frac{1}{p-1}}dx\right)^{p-1}.$$
$$\Rightarrow [\tau^z(w)]_{A_p} = Sup_{Q \text{ in } \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x - z)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x-z)^{-\frac{1}{p-1}}dx\right)^{p-1}$$
Put, $x - z = y \Rightarrow dx = dy$
$$\therefore [\tau^z(w)]_{A_p} = Sup_{Q \text{ in } \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(y)\,dy\right)\left(\frac{1}{|Q|}\int_Q w(y)^{-\frac{1}{p-1}}dy\right)^{p-1} = [w]_{A_p}.$$

Property 3.2
If $w \in A_p$, then the function $\lambda w \in A_p$. Moreover, $[\lambda w]_{A_p} = [w]_{A_p}$.
Proof: We have,
$$[\lambda w]_{A_p} = Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q (\lambda w)\,dx\right)\left((\lambda w)^{-\frac{1}{p-1}}dx\right)^{p-1}$$
$$\Rightarrow [\lambda w]_{A_p} = Sup_{Q\ in\ \mathbb{R}^n}\frac{1}{\lambda}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\lambda^{-1}\left(w(x)^{-\frac{1}{p-1}}dx\right)^{p-1}$$
$$\Rightarrow [\lambda w]_{A_p} = Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(w(x)^{-\frac{1}{p-1}}dx\right)^{p-1}$$
$$\Rightarrow [\lambda w]_{A_p} = [w]_{A_p}.$$

Property 3.3
If $\in A_p$, for some $1 \le p < \infty$ and non-negative measurable function k such that k, k^{-1} are in $L^\infty(\mathbb{R}^n)$ then $kw \in A_p$.
Proof: We have,
$$\left(\frac{1}{|Q|}\int_Q (kw)(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q (kw)\,(x)^{-\frac{1}{p-1}}dx\right)^{p-1}$$
$$= \left(\frac{1}{|Q|}\int_Q k(x)(w)(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q k(x)^{-\frac{1}{p-1}}(w)\,(x)^{-\frac{1}{p-1}}dx\right)^{p-1}$$
$$\le \left(\frac{1}{|Q|}\int_Q \|k\|_{L^\infty} w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \|k^{-1}\|_{L_\infty}^{\frac{1}{p-1}} w(x)^{-\frac{1}{p-1}}dx\right)^{p-1}$$
$$\le \|k\|_{L^\infty}\|k^{-1}\|_{L^\infty}[w]_{A_p} < \infty$$
$$\Rightarrow Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q (kw)(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q (kw)\,(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty.$$

Property 3.4
If $w_1,\ w_2 \in A_1$, then $w_1 w_2^{1-p} \in A_p$.
Proof: Let, $w_1, w_2 \in A_1$. Then by definition there exists a constant C such that
$Mw_i(x) \le Cw_i(x),\ i = 1, 2.$
$$\Rightarrow \frac{1}{|Q|}\int_Q w_i(x)\,dx \le [w_i]_{A_1} w_i(x) \Rightarrow \frac{1}{w_i(x)} \le [w_i]_{A_1}\frac{|Q|}{\int_Q w_i(x)\,dx}$$
$i.e.\ \frac{1}{w_i(x)} \le [w_i]_{A_1}\frac{|Q|}{w_i(Q)}$ for $x \in \mathbb{R}^n\ a.e.$
Now,
$$\frac{1}{|Q|}\int_Q w_1(x)\,w_2(x)^{1-p}dx = \frac{1}{|Q|}\int_Q w_1(x)\left(\frac{1}{w_2(x)}\right)^{p-1}dx \le [w_2]_{A_1}^{p-1}\left(\frac{|Q|}{w_2(Q)}\right)^{p-1}\frac{1}{|Q|}\int_Q w_1(x)\,dx = [w_2]_{A_1}^{p-1}\left(\frac{|Q|}{w_2(Q)}\right)^{p-1}\frac{w_1(Q)}{|Q|}$$
Also,
$$\left(\frac{1}{|Q|}\int_Q (w_1(x)w_2(x)^{1-p})^{-\frac{1}{p-1}}dx\right)^{p-1} = \left(\frac{1}{|Q|}\int_Q w_1(x)^{-\frac{1}{p-1}}w_2(x)\,dx\right)^{p-1}$$
$$= \left(\frac{1}{|Q|}\int_Q \left(\frac{1}{w_1(x)}\right)^{\frac{1}{p-1}} w_2(x)\,dx\right)^{p-1}$$
$$\le [w_1]_{A_1}\frac{|Q|}{w_1(Q)}\left(\frac{1}{|Q|}\int_Q w_2(x)\,dx\right)^{p-1} = [w_1]_{A_1}\frac{|Q|}{w_1(Q)}\left(\frac{w_2(Q)}{|Q|}\right)^{p-1}$$
Thus we have,

$$\left(\frac{1}{|Q|}\int_Q w_1(x)\,w_2(x)^{1-p}dx\right)\left(\frac{1}{|Q|}\int_Q (w_1(x)w_2(x)^{1-p})^{-\frac{1}{p-1}}dx\right)^{p-1} \le [w_1]_{A_1}[w_2]_{A_1}^{p-1} < \infty.$$

Property 3.5

If $w \in A_p$ and $a \in \mathbb{R}, a \neq 0$, then the function $\delta_a w \in A_p$, where $\delta_a w(x) = w(ax)$. Moreover, $[w]_{A_p} = [\delta_a w]_{A_p}$.

Proof: We have,

$$Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(ax)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(ax)^{-\frac{1}{p-1}}\,dx\right)^{p-1}$$

$$= Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(y)\frac{dy}{a}\right)\left(\frac{1}{|Q|}\int_Q w(y)^{-\frac{1}{p-1}}\,dy/a^{-\frac{1}{p-1}}\right)^{p-1}$$

$$= Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}$$

Hence the property follows.

Property 3.6

$$\left[w^{-\frac{1}{p-1}}\right]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}.$$

Proof: We have,

$$[w]_{A_p}^{\frac{1}{p-1}} = \left[Sup_Q \langle w\rangle_Q \left\langle w^{-\frac{1}{p-1}}\right\rangle_Q^{p-1}\right]^{\frac{1}{p-1}}$$

$$= Sup_Q \langle w\rangle_Q^{\frac{1}{p-1}} \left\langle w^{-\frac{1}{p-1}}\right\rangle_Q$$

$$= Sup_Q \left\langle w^{-\frac{1}{p-1}}\right\rangle_Q \left\langle \left(w^{-\frac{1}{p-1}}\right)^{-\frac{1}{p'-1}}\right\rangle_Q^{p'-1}$$

$$\Rightarrow\ [w]_{A_p}^{\frac{1}{p-1}} = \left[w^{-\frac{1}{p-1}}\right]_{A_{p'}}$$

Property 3.7

$[w]_{A_p} \geq 1$ for all $w \in A_p$. Equality holds if and only if w is a constant.

Proof: We have,

$$|Q| = \int_Q w(x)^{\frac{1}{p}}\left(\frac{1}{w(x)}\right)^{\frac{1}{p}} dx$$

$|Q| \leq \left(\int_Q w(x)^{\frac{1}{p}p}\,dx\right)^{\frac{1}{p}}\left(\int_Q \left(\frac{1}{w(x)}\right)^{\frac{1}{p}p'}\,dx\right)^{\frac{1}{p'}}$, using Hӧlder's inequality, we have

$$= \left(\int_Q w(x)\,dx\right)^{\frac{1}{p}}\left(\int_Q \left(\frac{1}{w(x)}\right)^{\frac{1}{p-1}} dx\right)^{\frac{1}{p'}}$$

$$\Rightarrow\ 1 \leq \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)^{\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q \left(w(x)\right)^{-\frac{1}{p-1}}dx\right)^{\frac{1}{p'}}$$

$$\Rightarrow\ 1 \leq Sup_Q\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{\frac{p}{p'}}$$

$$\Rightarrow\ 1 \leq Sup_Q\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}$$

Hence, $[w]_{A_p} \geq 1$.

Again we have, $1 = \frac{1}{|Q|}\int_Q dx$

$$\Rightarrow 1 = \frac{1}{|Q|}\int_Q w(x)^{\frac{1}{p}} w(x)^{-\frac{1}{p}}\,dx$$

Equality holds in Hölder's inequality only when $\alpha|f|^p = \beta|g|^{p'}$ for any constants α, β.

$$\Leftrightarrow \alpha\left|w^{\frac{1}{p}}\right|^p = \beta\left|w^{-\frac{1}{p}}\right|^{p'}$$

$$\Leftrightarrow \alpha w = \beta w^{-\frac{p'}{p}}$$

$$\Leftrightarrow w = \left(\frac{\beta}{\alpha}\right)^{\frac{p}{p+p'}}$$, which is a constant.

Property 3.8

The classes A_p are increasing as p increases; precisely for $1 \leq p < \infty$ we have

$A_p \subset A_q$, and $[w]_{A_q} \leq [w]_{A_p}$.

Proof: We have, $1 \leq p < q < \infty$. Since, $p < q \Rightarrow\ 1 - \frac{1}{p} < 1 - \frac{1}{q} \Rightarrow \frac{1}{p'} < \frac{1}{q'} \Rightarrow\ q' < p' \Rightarrow q' - 1 < p' - 1.$

$$\Rightarrow \|w^{-1}\|_{L^{q'-1}\left(Q,\frac{dx}{|Q|}\right)} \leq \|w^{-1}\|_{L^{p'-1}\left(Q,\frac{dx}{|Q|}\right)}$$

$$\Rightarrow \left(\int_Q (w^{-1})^{q'-1}\frac{dx}{|Q|}\right)^{\frac{1}{q'-1}} \leq \left(\int_Q (w^{-1})^{p'-1}\frac{dx}{|Q|}\right)^{\frac{1}{p'-1}}$$

$$\Rightarrow \left(\frac{1}{|Q|}\int_Q (w^{-1})^{q'-1}\,dx\right)^{\frac{1}{q'-1}} \leq \left(\frac{1}{|Q|}\int_Q (w^{-1})^{p'-1}\,dx\right)^{\frac{1}{p'-1}}$$

$$\Rightarrow \left(\frac{1}{|Q|}\int_Q (w^{-1})^{\frac{1}{q-1}}\,dx\right)^{q-1} \leq \left(\frac{1}{|Q|}\int_Q (w^{-1})^{\frac{1}{p-1}}\,dx\right)^{p-1}$$

Which implies,

$$Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q (w^{-1})^{\frac{1}{q-1}}\,dx\right)^{q-1}$$

$$\leq Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q (w^{-1})^{\frac{1}{p-1}}\,dx\right)^{p-1}$$

$$\therefore [w]_{A_q} \leq [w]_{A_p}.$$

Property 3.9

If $p > 1$ then $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$, where p' is the conjugate exponent of p such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof: " $\Rightarrow$ " We have,

$$\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\right)\left(\frac{1}{|Q|}\int_Q \left(w(x)^{1-p'}\right)^{-\frac{1}{p'-1}}dx\right)^{p'-1}$$

$$= \left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)^{\frac{1}{p-1}}$$

$$= \left[\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\right]^{\frac{1}{p-1}}$$

$$\leq \left[Sup_Q\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\right)^{p-1}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\right]^{\frac{1}{p-1}}$$

$$= \left([w]_{A_p}\right)^{\frac{1}{p-1}}$$

This shows that $w(x)^{1-p'} \in A_{p'}$.

" $\Leftarrow$ " Conversely assume that $w(x)^{1-p'} \in A_{p'}$ with $1 < p < \infty$. We have,

$$\left[\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}\right]^{\frac{1}{p-1}}$$

$$= \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)^{\frac{1}{p-1}}\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)$$
$$= \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)^{p'-1}\left(\frac{1}{|Q|}\int_Q \left(w(x)\right)^{1-p'}\,dx\right)$$
$$= \left[\frac{1}{|Q|}\int_Q \left(w(x)^{1-p'}\right)^{-\frac{1}{p'-1}}\,dx\right]^{p'-1}\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\right)$$
$$\leq Sup_Q\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\right)\left[\frac{1}{|Q|}\int_Q \left(w(x)^{1-p'}\right)^{-\frac{1}{p'-1}}\,dx\right]^{p'-1}$$
$$= \left[w(x)^{1-p'}\right]_{A_{p'}} \Rightarrow w \in A_p$$

Property 3.10

Let, $v,\ w \in A_p$ for $1 < p < \infty$. Then, $max\{v(x), w(x)\} \in A_p$.

Proof: Since, $v,\ w \in A_p, max\{v(x), w(x)\}$ is a weight. We have,

$$\frac{1}{|Q|}\int_Q max\{v(x), w(x)\}\,dx \leq \frac{1}{|Q|}\int_Q v(x)\,dx + \frac{1}{|Q|}\int_Q w(x)\,dx$$
$$= \frac{1}{|Q|}\int_Q v(x)\,dx\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}\,dx\right)^{1-p}$$
$$+ \frac{1}{|Q|}\int_Q w(x)\,dx\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{1-p}$$
$$\leq [v]_{A_p}\left(\frac{1}{|Q|}\int_Q v(x)^{-\frac{1}{p-1}}\,dx\right)^{1-p}$$
$$+ [w]_{A_p}\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{1-p}$$
$$\leq \left([v]_{A_p} + [w]_{A_p}\right)\left(\frac{1}{|Q|}\int_Q max\{v(x), w(x)\}^{-\frac{1}{p-1}}\,dx\right)^{1-p}$$
$$\Rightarrow \frac{1}{|Q|}\int_Q max\{v(x), w(x)\}\,dx\left(\frac{1}{|Q|}\int_Q max\{v(x), w(x)\}^{-\frac{1}{p-1}}\right)^{p-1}$$
$$\leq [v]_{A_p} + [w]_{A_p}$$

Thus, $max\{v(x), w(x)\} \in A_p$.

Property 3.11.

$\lim_{q\to 1+}[w]_{A_q} = [w]_{A_1}$ if $w \in A_1$.

Proof: A weight w is said to be of class A_q if

$$[w]_{A_q} := Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{q-1}}\,dx\right)^{q-1} < \infty$$
$$\Rightarrow [w]_{A_q} = Sup_{Q\ in\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\frac{1}{|Q|^{q-1}}\left(\int_Q w(x)^{-\frac{1}{q-1}}\,dx\right)^{q-1}$$
$$\Rightarrow [w]_{A_q} = Sup_{Qin\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\frac{1}{|Q|^{q-1}}\left(\int_Q (w(x)^{-1})^{\frac{1}{q-1}}\,dx\right)^{q-1}$$
$$\Rightarrow [w]_{A_q} = Sup_{Qin\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\frac{1}{|Q|^{q-1}}\|w^{-1}\|_{L^{\frac{1}{q-1}}(Q,dx)}$$

When $q \to 1+$, we have

$$[w]_{A_q} = Sup_{Qin\ \mathbb{R}^n}\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\frac{1}{|Q|^{q-1}}\|w^{-1}\|_{L^{\infty}(Q,dx)}$$
$$\therefore \lim_{q\to 1+}[w]_{A_q} = [w]_{A_1}.$$

Property 3.12.

An equivalent characterization of A_p constant of w is

$$[w]_{A_p} = Sup_{Q\ in\ \mathbb{R}^n} Sup_{f\ in\ L^p(Q,wdx),|f|>0\ a.e.on\ Q}\left\{\frac{\left(\frac{1}{|Q|}\int_Q|f(t)|dt\right)^p}{\frac{1}{w(Q)}\int_Q|f(t)|^p w(t)dt}\right\}.$$

Proof: We have,

$$\left(\frac{1}{|Q|}\int_Q|f(x)|\,dx\right)^p = \left(\frac{1}{|Q|}\int_Q|f(x)|w(x)^{\frac{1}{p}}w(x)^{-\frac{1}{p}}\,dx\right)^p$$

Applying Hölder's inequality for the functions $|f(x)|^p w(x)^{\frac{1}{p}}$ and $w(x)^{-\frac{1}{p}}$ with conjugate exponents $p,\ p'$,

$$\left(\frac{1}{|Q|}\int_Q|f(x)|\,dx\right)^p \leq \frac{1}{|Q|^p}\left[\left(\int_Q\left(|f(x)|w(x)^{\frac{1}{p}}\right)^p dx\right)^{\frac{1}{p}}\left(\int_Q\left(w(x)^{-\frac{1}{p}}\right)^{p'} dx\right)^{\frac{1}{p'}}\right]^p$$
$$= \frac{1}{|Q|^p}\left(\int_Q|f(x)|^p\, w(x)dx\right)\left(\int_Q w(x)^{-\frac{p'}{p}}\,dx\right)^{\frac{p}{p'}}$$
$$= \frac{1}{|Q|^p}\left(\int_Q|f(x)|^p\, w(x)dx\right)\left(\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}$$
$$\Rightarrow \left(\frac{1}{|Q|}\int_Q|f(x)|\,dx\right)^p \leq \left(\frac{1}{w(Q)}\int_Q|f(x)|^p\, w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \leq \frac{1}{w(Q)}\int_Q|f(x)|^p w(x)\,dx\,[w]_{A_p}$$
$$\Rightarrow \frac{\left(\frac{1}{|Q|}\int_Q|f(x)|\,dx\right)^p}{\frac{1}{w(Q)}\int_Q|f(x)|^p\, w(x)dx} \leq [w]_{A_p} \quad \ldots \ldots (1)$$

Also set $f = (w+\varepsilon)^{-\frac{p'}{p}}$ in $Sup_{Q\ in\ \mathbb{R}^n} Sup_{f\ in\ L^p(Q,wdx),|f|>0\ a.e.on\ Q}\left\{\frac{\left(\frac{1}{|Q|}\int_Q|f(t)|dt\right)^p}{\frac{1}{w(Q)}\int_Q|f(t)|^p w(t)dt}\right\}$, we have

$$Sup_{Q\ in\ \mathbb{R}^n} Sup_{f\ in\ L^p(Q,wdx),|f|>0\ a.e.on\ Q}\left\{\frac{\left(\frac{1}{|Q|}\int_Q(w+\varepsilon)(t)^{-\frac{p'}{p}}\,dt\right)^p}{\frac{1}{w(Q)}\int_Q(w+\varepsilon)(t)^{-p'}w(t)\,dt}\right\}$$
$$= Sup_{Q\ in\ \mathbb{R}^n} Sup_{f\ in\ L^p(Q,wdx),|f|>0\ a.e.on\ Q}\frac{1}{|Q|}w(Q)\left(\frac{1}{|Q|}\int_Q(w+\varepsilon)^{-\frac{p'}{p}}(t)\,dt\right)^p\left(\frac{1}{|Q|}\int_Q\frac{w(t)}{(w+\varepsilon)(t)^{p'}}\,dt\right)^{-1}$$
$$\geq Sup_{Q\ in\ \mathbb{R}^n} Sup_{f\ in\ L^p(Q,wdx),|f|>0\ a.e.on\ Q}\frac{1}{|Q|}w(Q)\left(\frac{1}{|Q|}\int_Q(w+\varepsilon)^{-\frac{p'}{p}}(t)\,dt\right)^p\left(\frac{1}{|Q|}\int_Q\frac{(w+\varepsilon)(t)}{(w+\varepsilon)(t)^{p'}}\,dt\right)^{-1}$$
$$= Sup_{Q\ in\ \mathbb{R}^n} Sup_{f\ in\ L^p(Q,wdx),|f|>0\ a.e.on\ Q}\frac{1}{|Q|}w(Q)\left(\frac{1}{|Q|}\int_Q(w+\varepsilon)(t)^{-\frac{p'}{p}}\,dt\right)^p\left(\frac{1}{|Q|}\int_Q(w+\varepsilon)(t)^{1-p'}\,dt\right)^{-1}$$
$$= Sup_{Q\ in\ \mathbb{R}^n} Sup_{f\ in\ L^p(Q,wdx),|f|>0\ a.e.on\ Q}\frac{1}{|Q|}\int_Q w(t)\,dt\left(\frac{1}{|Q|}\int_Q(w+\varepsilon)(t)^{-\frac{1}{p-1}}\,dt\right)^{p-1}$$
$$= [w]_{A_p}, \text{ when } \varepsilon \to 0.$$

$$\Rightarrow Sup_{Q\ in\ \mathbb{R}^n} Sup_{f\ in\ L^p(Q,wdx),|f|>0\ a.e.on\ Q}$$

$$\left\{\frac{\left(\frac{1}{|Q|}\int_Q (w+\varepsilon)(t)^{-\frac{p'}{p}}\,dt\right)^p}{\frac{1}{w(Q)}\int_Q (w+\varepsilon)(t)^{-p'} w(t)\,dt}\right\} \geq [w]_{A_p} \quad \dots\dots \quad (2)$$

From inequality (1) and (2) the property follows.

Property 3.13

The measure $w(x)dx$ is doubling, precisely for all $\lambda > 1$ and all cubes Q we have $w(\lambda Q) \leq \lambda^{np}[w]_{A_p} w(Q)$.

Proof: Set $f = \chi_Q$ and λQ for Q in the inequality

$$\frac{\left(\frac{1}{|Q|}\int_Q |f(t)|\,dt\right)^p}{\frac{1}{w(Q)}\int_Q |f(t)|^p w(t)\,dt} \leq [w]_{A_p}$$

we have $\frac{\left(\frac{1}{|\lambda Q|}\int_Q \chi_Q dt\right)^p}{\frac{1}{w(\lambda Q)}\int_Q \lambda_Q w(t) dt} \leq [w]_{A_p}$

$$\Rightarrow \frac{\frac{1}{|\lambda Q|^p}\left(\int_Q \chi_Q\,dt\right)^p}{\frac{1}{w(\lambda Q)}\int_Q \lambda_Q w(t)\,dt} \leq [w]_{A_p} \Rightarrow \frac{w(\lambda Q)\left(\int_Q \chi_Q\,dt\right)^p}{\int_Q \chi_Q w(t)\,dt}$$

$$\leq [w]_{A_p}|\lambda Q|^p \Rightarrow w(\lambda Q)|Q|^p$$

$$\leq [w]_{A_p}|\lambda Q|^p w(Q) \Rightarrow w(\lambda Q)|Q|^p$$

$$\leq [w]_{A_p}\lambda^{np}|Q|^p w(Q) \Rightarrow w(\lambda Q)$$

$$\leq [w]_{A_p}\lambda^{np} w(Q).$$

4. CONCLUSION

We revisit the literature on weighted inequalities and discuss some properties of A_p weight functions with proof that will enhance the understanding of weight theory and set the course for further study and research in this domain.

CONFLICT OF INTEREST

No any conflict 0f interest.

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