Article DOI: **https://doi.org/3126/ajme.v8i1.95324**

Academic Journal of Mathematics Education

A Double-blind Peer Reviewed Journal

ISSN 2645-8292; Volume 8, Issue 1, December 2025, pp. 69-84

Indexed in Nepal Journals Online (NepJOL)

Innovative Pedagogical Plan to Conceptualize the Axiomatic Concept of Group Theory

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Abstract

This innovative plan was thought to address the problem of conceptualization of group theory by undergraduate students. The design-based research was used to design, test, update, and finalize the innovation. The action process, object, and schema (APOS) theory was used to design innovation by following the sequence of inquiry-oriented instruction, a type of inquiry-based learning. The research utilized forms and formats of design-based research. Before starting definition of groups, different properties of a set and an operation are explored. Students create different examples of groups themselves in the given condition and extend the created groups by adding minimal elements. Students test a list of sets and operations to conceptualize group. As a result, students can use their own symbols of group axioms. The finalized plan is a ready to use resource to faculty members in helping students to conceptualize the group theory at undergraduate level. Based on the result of the quiz, classroom observation, interview, it is reflected that the concept by construction strategy is best for conceptual understanding, mathematical thinking, and creativity. The developed innovation plan is a reference resource for teachers who follow inquiry-based learning in their classroom. This plan has served as a best tool for building concept of group theory because students got opportunity to act as mathematicians.

Article Info.

Article History

Received: 16 October 2025

Revised: 2 December 2025

Accepted: 6 December 2025

Publisher

Council for Mathematics Education Branch, Surkhet

Keywords: APOS theory, inquiry-oriented instruction, design-based research, undergraduate students, abstract algebra

Introduction

Undergraduate level is considered the first and foundation level of higher education (Saxe & Braddy, 2015) as well as professional prepartion. Three undergraduate programs of Nepal namely Bachelor of Arts (B A), Bachelor of Science (B Sc), and Bachelor of Education (B Ed) offer mathematics courses for students who aspire to develop career related to mathematics. Students study the axiomatic mathematics for the first time when they study abstract algebra. It is expected that students typically spend their class time working on mathematical tasks and sharing their findings in some way (Fukawa-Connelly, 2007). Group theory is the first axiomatic concept taught in abstract algebra at the undergraduate level. Thus, teaching group theory at undergraduate level requires several considerations as they learn through intuition, induction, contrast, analogy, simulation, and analysis (Zaffar et al., 2013). Learning abstract algebra requires active methods of learning rather than formal lecture.

Concepts are considered fundamental learning elements in mathematics. Learning concepts in group theory is not trivial. Understanding concepts is considered a milestone for the advanced level of mathematics. Also, understanding concepts is necessary for a high degree of success in proving theorems (Melhuish, 2015). The constructivist approach is considered the best way to deliver generalization and abstraction in mathematics (Mitchelmore, 2002). Sound conceptual understanding is highly required to have mastery of abstract algebra (Risnanosanti, 2018).

Instruction process is considered to be the empowering factor for enhancing mathematical skills. The facilitation process is expected to be excelling mathematical skills as well as develop positive image of mathematics in students. Though it is expected to be empowered, it also has impacted negative image towards mathematics. Since long time, there were very few of the students who had positive images of mathematics (Gray & Tall, 1993). In recent time, in Nepal, undergraduate students perceive mathematics as applicable in different fields, thought they also perceive it as a difficult and abstract, as decontextualized, and as mysterious subject (Lamichhane & Belbase, 2017). The curricular and pedagogical approaches, the conventional and disempowering methods of teaching, used from school to university level are the main influencing factors for those negative images (Lamichhane & Belbase, 2017).

It is suggested that the central focus of the group theory is the understanding of closure property, associative property, identity, inverse, function, and the structure of set (Arnawa et al., 2007). This shows that there is a problem with directly moving to the definition of group so that students can't focus on different properties of a set with an operation.

To address the problem an innovative approach of building concept namely the concept by construction was developed. This study mainly focused on using innovations: Concept by construction strategies, in the learning of the fundamental concept of group theory in an axiomatic approach in undergraduate mathematics education programs. Apart from improving the innovation, the main objective of the study was to explore the ways undergraduate students build learning trajectories in learning group theory by following hands-on inquiry-oriented instruction.

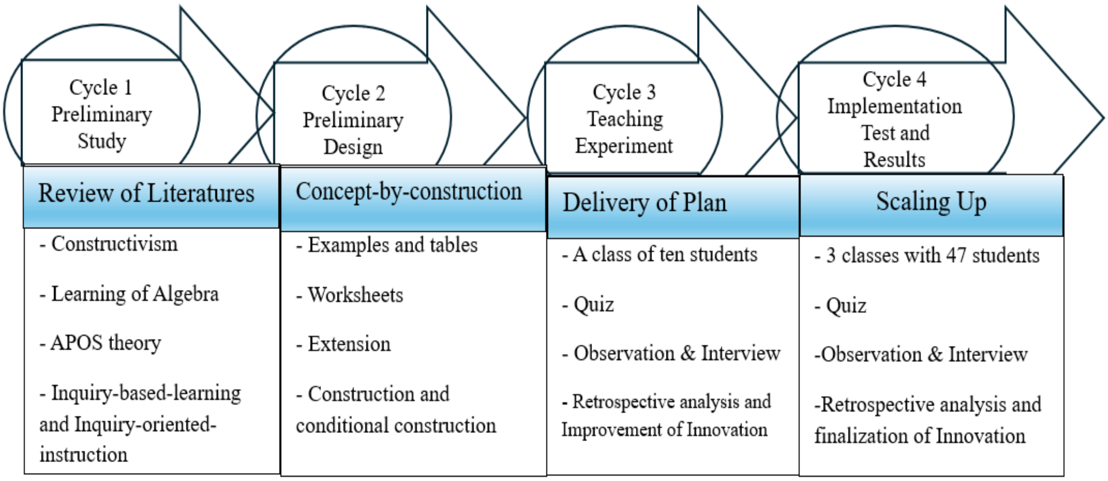
Research Question

The research question for the study was "How can the learning of concept of group theory be improved by using the concept by construction strategies at undergraduate level?".

This paper follows the sequence as presented in Figure 1 in cyclical approach. The first cycle is preliminary study. The solution to the problem was thought of from the constructivist approach of learning. The second cycle is the preliminary design: the design of innovative lessons following the theoretical framework. The third cycle is to test in the small classroom, and finally, the fourth cycle is related to scaling up a larger group of students and finally the innovative learning trajectory was updated.

Figure 1

Design Framework of Concept-by-Construction Strategy in Group Theory



Review of Literatures

There are different schools of thought on how learning occurs. Constructivist learning is one of them. According to constructivism, learners actively construct new knowledge pursuant to his/her existing knowledge (Ah-Nam & Osman, 2017). Constructivists advocate active learning. Active learning is meant for engaging students into activities which promote higher order thinking (Braun et al., 2017). Nine guiding principles are used to apply constructivism in the classroom. First, learning is an active process. Second, people learn to learn as they learn. Third the crucial action of constructing meaning is mental. Fourth, learning involves language. Fifth, learning is a social activity. Sixth, learning is contextual. Seventh, one needs knowledge to learn. Eighth, it takes time to learn. Ninth, motivation is the key component in learning (Hein, 1991). Letting students learn via inquiry is one of the ways constructivists mostly advocate.

Inquiry-based learning (IBL) is an approach of learning, where students learn by the help of inquiry questions. Generally, scholars believe IBL is suitable for science, but IBL is also compatible with learning mathematics (Caswell & LaBrie, 2017). In IBL mathematics, teachers provide a good sequence of tasks, and a positive and supportive group dynamic so that students can experience both successes and failures, each in a measure appropriate for fostering their growth as mathematicians (Ernst et al., 2017). Inquiry-oriented instruction (IOI), a particular form of IBL, moves on generating students' ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation (Kuster et al., 2018). Thus, IOI utilizes carefully designed and sequenced resources to share thinking with co-learners and engages co-learners' thinking. In summary, IOI is directed by two goals namely students who share their thinking and students engage in other's thinking (Kuster et al., 2018).

Action, process, object, and schema (APOS) theory is used to describe the level of mathematical understanding of a person with the help of mental mechanisms. APOS theory is described through its two components: mental structures or stages and mental mechanism. Action, process, objects, and schema are mental stages. Interiorization, coordination, reversal, encapsulation, and thematization are five types of mental mechanisms. Thus, according to APOS theory, a concept is first conceived as an Action, that is, as an externally directed transformation of a previously conceived object(s). In the action phase each step of transformation requires explicit and externally guided instructions. When the material action is reconstructed in the mind of the subject, which is the internal mental construction of the material action to become process (Arnon, et al., 2014). The mental construction that turns action concept to process conception is known as 'interiorization' (Arnawa et al., 2007). When the learner interiorizes the action in his/her mind without all the specific steps of algorithm, it becomes a process (Clark, et al., 1997). The learner now can use this process to obtain new process either through coordination or reversal. When process concept is transformed by some actions, then it is said to be encapsulated to become an object (Clark, et al., 1997). The mathematical object exists when it is encapsulated in an individual's mind which requires as assignment of label to the object (Dubinsky, et al., 1994). The evocated actions, processes, objects, and other schemas to deal with a new problem situation is jointly a schema.

Theoretical analysis describes action, process, object, and schema that students might construct when developing a mathematical concept (Arnawa et al., 2007). Pedagogical analysis based on inquiry-based instruction, the innovative idea of concept by construction and hands-on toolkits were additional major considerations in designing an innovative plan. Activities and exercises are considered as the main mechanism to help students to construct actions, interiorize into the processes, encapsulate into object (Arnon, et al., 2014). This research used the activities guided by the principles of Inquiry-oriented instruction (IOI) to observe the learning of mathematics at undergraduate level.

By reviewing literature and theory, an innovative solution namely concept-by-construction by following the steps of IOI was initially designed to address the problem of conceptualization of group theory.

Research Methods

An instructional plan was designed to address the problem of learning axiomatic concept of group theory. This design-based research (DBR) was planned and implemented in a regular college class. This DBR was designed to explore the systems of learning group theory with a focus on what students did and how students learned. This research has 'an explanatory and advisory aim' (Doorman, 2019, p. 4) on how innovative ways of teaching group theory. The designed context was 'emergent' and it involves 'Iterative cycle of designing' (Henrick et al., 2015). Design-based research has a cycle of three phases: the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis. The study followed design-based research by following the steps: Preliminary design, implementation of design, retrospective analysis and finalization of the design.

Preliminary Design of Concept by Construction

Concept by construction is an innovation developed and tested and refined during the study. In the strategy, students are asked to construct examples of group structures as per the given requirements. This part of innovation consists of three approaches to conceptualize the group. First, before defining the group, different properties of set and operation are explored, so that the exploration helps conceptualize the group. Second, the creation of groups by adding some elements so that students can play with different properties required for a group. This leads students to construct groups of their own. Third, students test whether a set with an operation formed a group or not. The creation of own way of defining group axioms is also an additional task added after the first teaching experiment.

The initial innovation prototype was presented to master's degree class. Based on the feedback received on the presentation to the class, the prototype was again refined and presented in a Ph. D. seminar. Based on the feedback in the seminar, the prototype was further refined and made ready to implement it. Initially the prototype was designed for four hours and an hour of class-quiz. In order to finalize the plan, the researchers planned to collect evidence for further improvement in real-time situation, and we followed the process of design-based research.

Research Instruments and Process

The research was mainly based on how students develop learning trajectories in the designed innovation on learning the concept of group theory. The innovation plan was one of the main instruments in the research. There were three worksheets used during the research process. The researchers used some of the guiding questions for process interviews. The principal researcher also prepared his notes based on the observation and interview. There was a quiz administered after the implementation of the innovation.

Students were provided with a written information sheet to read by themselves. An oral explanation was also made before beginning the research process. The participants were provided with a note of confidentiality stating that the d ata provided bythem or obtained from them wouldbe kept .confidentialThe researchers followed ethical guidelines as mentioned in the approval certificate of the Institutional Review Board (IRB) of Mahidol University (COA No. 2019/12-472).

Implementation: Teaching Experiment

The innovation plan of concept by construction was implemented to a class of undergraduate students, who registered abstract algebra for the first time. There were ten students, and they were divided into three groups having 3, 3, and 4 members in each. The innovation was planned for four hours of learning. But it took six hours and two hours each day. An interview with three students in each group was conducted around two hours for each. After completion implementation, there was a quiz for assessing students' learning for about one hour. After the retrospective analysis of the first phase, the plan was revised. Before the scaling up, a workshop with the three faculty members was organized. Based on the workshop feedback the draft was revisited and finalized for the implementation. Then again it was scaled up to three colleges where there were 21, 14 and 12 students totaling 47 students. Each class was divided into groups having 3 to 4 members.

Retrospective Analysis

The retrospective analysis is the systematic and reflective process of examining dataset, generated during the teaching experiment phase, to decide how and why learning occurred or didn't occur. The techniques of argumentative grammar was used which lead the concerns about the warrant for claims (Cobb et al., 2015). How each subsequent form of reasoning developed was documented. The elements of the classroom learning environment that facilitated the students' reasoning were identified. Based on the retrospective analysis, in the teaching experiments, the innovation plan is finalized as presented herewith.

Improved Innovation Plan

In concept by construction, students are asked to construct a structure of group by using the conditions or following the given instructions. The following learning trajectory was revised and finalized after the cycles of implementations. The classroom practices are planned in day-wise sequence of two hours each. Each day consists of a question of inquiry, which is a question or instruction for an activity to be expected to achieve at the end, learning trajectory, and sequence of tasks instruction or guiding questions.

Day 1 Properties of a Set and an Operation

This day is planned to build foundations for the conceptualization of the algebraic structure of group theory. Specifically, the main intention of the plan is to help learners in exploring the properties of a set and an operation. The design of the plan is guided by the action phase and transfer to the process phase of the APOS theory.

Inquiry Question 1

Write a set and define an operation on it and list the properties satisfied/unsatisfied by the operation on the set.

Learning Trajectories

Learning consists of some of the experienced learning trajectories of the inquiry question in the process. Students are expected to use a variety of sets and operations. Students mostly start with a finite set, and usual operations: addition, subtraction, multiplication, and division. It can be expected that students test almost all the properties: closure, associative, commutative, identity, and inverse. Later, students think of some well-known set of numbers such as the set of natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), and rational numbers ($\mathbb{Q}$).

Guiding Questions/Instructions

The learning trajectory can be guided by questions or instructions for activities as mentioned in the following sequence.

1) Make a list of properties: This can be done with the question, “What properties do you know about a set and an operation?” This question may need to be further supported by giving clues related to properties or asking another question: “Do you know property like commutative property, identity, etc.?” to some groups. It is expected to make a list of those properties required to learn the concept of group theory. That is, closure, associative, commutative, identity, and inverse. Students can give a try to give examples of such properties. It may not just names but can also be asked to explain these properties with examples. The students may or may not know the meanings of those properties.

2) Check a property is satisfied: It is not necessary to follow the sequence as described here. Students are asked to create a finite set for example A= {1, 2, 3, 4, 5} and select usual operations addition, subtraction, multiplication, division and test whether a property is satisfied. There can be a question, "What are the properties, you see satisfied and not satisfied by the operation in the set?". In retrospective analysis, not satisfied was added as students left some of the properties without testing as not satisfied. It was also added to provide a list of properties in a display so that they all can test one by one.

Students may take similar nature of sets such as $A = \{1, 2, 3,4, 5\}$ and $B = \{ 2, 4, 6, 8,10, 12\}$ so that they don't get a variety of properties. After retrospective analysis, they were to ask to create a variety example by asking them to create set and operation that satisfies all the properties. It was also added to provide clue for a property by asking a question- "When do we say that a set and an operation satisfy a closure property?" This type of question was asked if necessary. For example, 1) Set $\mathbb{N} = \{1, 2, 3, \dots\}$ and addition of natural numbers is the operation, is it closed? What about subtraction, multiplication, and division? 2) Given the set $\mathbb{Z} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, and the addition of integers is the operation. Is it closed? What about subtraction, multiplication, and division of integers? 3) Given the set $\mathbb{Z}_n = \{0, 1, 2, 3, \dots, n-1\}$ (n = 5, 6, 7 for example) with operation on addition and multiplication on $\mathbf{Z}_n$. Is the set closed with the operation? Retrospective analysis suggested to add that there should be metaphors such as passing in the exams to test and declare whether a property is satisfied or not. To pass an exam, students need to pass all subjects, similar to that to hold a property, it is necessary to satisfy in all cases. If fail in one subject, the result is failed. Similar to that if failed to satisfy in one case, the property is not satisfied.

3) Create Own Operation: The instruction: Write a set of your own. Create your operation and test whether the properties are satisfied or not. Initially, students may work on creating an operation that satisfies at least two/three properties. They may need an explanation of different types of operation by giving the following example: $X = \{1, -1\}$, define an operation $*$ as follows: $a * b = \frac{1}{2}(a^2 + b^2)$. Is it closed? Is it associative? … Students may only be limited to create similar types of sets and operations by making different powers, for example cube.

4) Meaning of Properties: Students can be asked to give formal meaning [definitions] to these properties. Initially, students can give meanings of these properties using addition and multiplication operations rather than the expectation of this task to define the properties using mathematical symbols. There can be an expectation that students can explain these properties in words rather than by using mathematical symbols. It can also be too early to generate a formal definition of these properties and can be decided to move, during the retrospective analysis, this task after they play with several other examples.

5) Worksheet for Consolidation: This is a consolidation of properties of set and operation by preparing a list of set and operation on the set. They are asked to fill up the worksheet related to the properties of set and operation as shown in Figure 2. There can be a clue to making a set and operation that satisfies as many properties as possible so that they come up with the sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R},$ finite set like $\{1, -1\}, \{0, 1\}, \mathbb{Z}_5,\ \mathbb{Z}_6$. It can be a guided question: "What properties are satisfied in a set (the set that is not considered by students) with an operation (the operation is not taken by students)?" to include different sets including $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and some finite sets such as $\{1, -1\}, \{0\}, \{0, 1, 2\}$ and $\mathbb{Z}_n$ for n = 3, n = 4, and n = 5. The guided question was added during the first cycle of implementation.

Figure 2

Worksheet I: Properties of Set and Operation

Set (X)	Operation (*)	Properties satisfied or not				
		Closed ______	Commutative ______	Associative ______	Identity ______	Inverse ______

6) Sharing in Whole Group: Finally, as an exit ticket, students are asked to share and discuss in the whole group what they learned on the day. Each group can show eagerness to make a presentation of what they had learned on the day and try to add interesting examples from another group.

Day 2 Creation of Groups by Construction

This day is devoted on creating groups under addition and multiplication operations by students themselves; that is a structure of a non-empty set with an operation satisfying closure, associative, identity, and inverse. The main intention of the activities is to develop the concept of group as an algebraic structure. These activities are helpful to develop the concept of a group so that students think from the process to object stage of the APOS theory. The activities designed here are linked with conditional construction.

Inquiry Question 2

Create groups by having minimal element(s) a) under addition and b) under multiplication and then extend by adding a minimum number of elements under the given operations.

Learning Trajectories

Before the task of creating a group, it can also be expected that students need to develop an understanding that a set having an identity element forms a group with a minimum number of elements. But it can be different for them when they are asked to create a group with minimum elements. It can be noticed that students start with an identity element, but they think of other elements to satisfy other axioms such as closure property and associative property. Finally, they create groups $\{0\}$ under addition and $\{1\}$ under multiplication as the group having minimum element. The main expectation of adding a minimum number of elements in each of the groups with respective groups is to conceptualize a group with a high level of thinking. Students are expected to create a variety of sets. Students can be asked to repeat the process of adding minimum elements until they come up with infinite sets.

Guiding Questions/Instructions

The following five sets of instructions are finalized after the analysis of the cycles implementation to guide students to form groups under the given condition.

1) Review of Group Axioms: A review of learning about the properties of a set and an operation in the previous day can be the first activity of the day. Then the concept of group can be introduced by classifying sets and operations mentioned in Worksheet I as shown in Figure 1 as a group and as not a group. Based on their examples, it can be noticed that all groups are abelian and the term commutative can be used for making the notion of group. They are asked to generate the meaning and structure of the commutative group from the examples. They easily can come up with the meaning of commutative group. They simply can mention that a group satisfies all the properties: closure, associative, identity, and inverse.

2) Create Group with Minimal Elements: Some groups of students generated a group under multiplication at first and some of them generated group under addition at first during the first cycle. This parallel version is the refined form as students can create any of them.

Task I is to form a group under multiplication operation with a minimum number of elements; a non-empty set having a minimum number of elements that satisfies the four conditions (closed, associative, identity, and inverse). Further, they can be asked to test for commutative property. There is an expectation that students will construct a set having a single element such as {10} without thinking about other properties. But there are also other cases from students that created a set of at least two elements a and b for closure and in other case students create a set with three elements a, b, and c for associative. Then students are directed to go through the definition of closure and associative with the concern of number of two or three elements in mind. Task II is to form a group in addition operation with a minimum number of elements. The expectations and considerations are same as in Task I. These two tasks are assigned together.

3) Adding Minimum to Form Group: The following two tasks, Task III and Task IV, are provided together so that students can think either. In thinking, one task may guide another.

Task III is to form a set by adding a minimum number of elements in the set they made in Task I that satisfies the four conditions under multiplication: Closure, associative, identity, and inverse. Task IV is to form a set by adding a minimum number of elements in the set they made in Task II that satisfies the four conditions under addition.

Students can work simultaneously with Task III and Task IV or one after another in any order. These are the main activities to conceptualize the structure of a group. Task III and Task IV are repetitively used until students generated a variety of groups and come up with infinite sets like $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. The teacher needs to consider two things in students' work: adding a minimum number of elements and satisfying the axioms of a group. If some constructions are not groups, they can be asked to recheck properties by considering the elements they added.

4) Create Group of Your Own: Students are asked to create an example of group themselves. That is, they are instructed to write their own non-empty set with their own defined operation so that it forms a group. This is creative work and took considerable time to create. Some of the groups can come with the set and operation defined in earlier phases. Some of the students give a try to create new groups but struggle a lot, but later they create groups having one or two elements rather than many (or infinite elements).

5) Sharing in Whole Group: The sharing of creation by each group provides different ways of thinking and feedback. Then they can be asked to summarize their creations related to group and non-group as review and reflection. Students in their group can summarize their work in a worksheet as given in Figure 3.

Figure 3

Worksheet II: List of Created Groups/Non-Groups

Set (X)	Operation (*)	Properties satisfied or not				Check Group or not
		Closed ______	Associative ________	Identity ______	Inverse ______	

They can revise by filling in the worksheets. Once completed by all the groups, they can share their learning of the day.

Day 3 Test: Group or Not a Group

This day is designed to develop a range of examples of groups. The range includes mostly the infinite subsets of real numbers with the usual operations of addition and multiplication. This also included $\mathbb{Z}_n$ under operations addition and multiplication in modulo n. The activities mentioned herewith are expected to develop the object concept and strengthen the schema of the axiomatic concept of the group concerning APOS theory.

Inquiry Question 3

Decide whether the given set with an operation forms a group or not. Further, if it is a group, decide whether the group is an abelian group or not. (Given set: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{Q} - \{0\}, \mathbb{R} - \{0\}$, $\mathbb{Z}_n$, $n = 4, 5, 6$ and 7; n$\mathbb{Z}$, $n = 2, 3, 4, 5$ and 6).

Learning Trajectories

Students are expected to test the four axioms of group examples by themselves. Students are expected to present examples to show that the structure is not a group. They can discuss based on the examples led by the generalizations. Based on the generalizations, they can create a formal axiomatic definition of group as a learning of this part of innovation.

Guiding Questions/Instructions

The following four sets of instructions are finalized after the analysis of the first cycle implementation to guide students to form groups under the given condition in the learning process.

1) Review of Group Axioms: The class can start with a revision of the work by asking them two questions in sequence “What is a group, and then what are the groups you have constructed so far?”. Then students prepare a list of examples of the groups that they created/ tested.

2) Worksheet to Test Whether the Structure Forms a Group: To simplify work a worksheet can be designed, there can be a list of set and operation, as given in Figure 4. The worksheet can be asked to be filled up individually, once they fill up, they can be asked to be pair and discuss the differences. The whole group discussion is designed for the next step.

Figure 4

Worksheet III: Test Groups/Non-Groups/Abelian Group

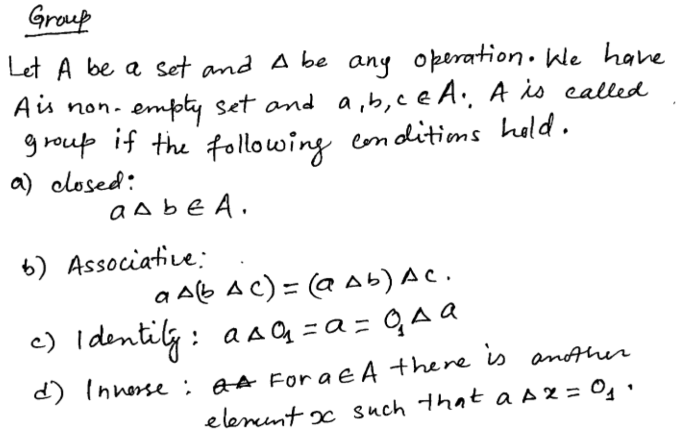
(Set, Operation)	Closed	Associative	Identity	Inverse	Group	Abelian
(Z, +)	✓	✓	✓	✓	✓	✓
(Z, .)	✓	✓	✓	✗	✗	✗
(Q, +)	✓	✓	✓	✓	✓	✓
(Q, .)	✓	✓	✓✗	✓✗	✓✗	✓✗
(R, +)	✓	✓	✓	✓	✓	✓
(R, .)	✓	✓	✓	✓	✓	✓
(2Z, +)	✓	✓	✓	✓	✓	✓
(3Z, +)	✓	✓	✓	✓	✓	✓
(4Z, +)	✓	✓	✓	✓	✓	✓
(2Z, .)	✓	✓	✓	✗	✗	✗
(3Z, .)	✓	✓	✓	✗	✗	✗
(4Z, .)	✓	✓	✓	✗	✗	✗
(Z_4, $+_4$)	✗	✗	✗	✗	✗	✗
(Z_5, $+_5$)	✗	✗	✗	✗	✗	✗
(Z_6, $+_6$)	✗	✗	✗	✗	✗	✗
(Q-{0}, .)	✓	✓	✓	✓	✓	✓
(R-{0}, .)	✓	✓	✓	✓	✓	✓
✗ (Z_5-{0}, $._5$)	✗	✗	✗	✗	✗	✗
✗ (Z_6-{0}, $._6$)	✗	✗	✗	✗	✗	✗

3) Discussion: Whole group discussion based on the worksheet can be divided into three parts: a) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, in addition, multiplication [Discussion needs to lead to make omit $\{0\}$ in case of multiplication. For example, $\mathbb{Q} - \{0\}$ is the group in multiplication. b) $\mathbb{Z}_n$, $n = 3, 4, 5, 6$, etc. addition and multiplication in mod n. c) $n\mathbb{Z}$, $n = 2, 3, 4, 5$ ... in addition and multiplication in respective modulo. It should be noted that students can come up to generate $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ in multiplication they did not consider the inverse of 0. The question of finding the multiplicative inverse of 0 is noticed to be helpful to realize that 0 has no multiplicative inverse.

4) Create Your Own Group Axioms: Students can be asked to create their definition of group, that is, their own way of writing group axioms, which is true for all sets and all operations. The variability of the set and the variability of operations can be generalized from the examples. Students can generate their own definitions rather than what was presented in the textbook. One sample creation is given in Figure 5 for reference. This can lead that students can work as mathematicians.

Figure 5

Students' Creation of Group Definition


Group

Let A be a set and Δ be any operation. We have A is non-empty set and $a, b, c \in A$. A is called group if the following conditions hold.

a) closed:
$a \Delta b \in A$.

b) Associative:
$a \Delta (b \Delta c) = (a \Delta b) \Delta c$.

c) Identity: $a \Delta O_1 = a = O_1 \Delta a$

d) Inverse: ~~a Δ~~ For $a \in A$ there is another element x such that $a \Delta x = O_1$.

Results and Discussion

A class quiz was administered as an assessment of learning the concept of the group at the end of the session. The first item was to define two operations on set A = {1} so that it gives two groups with the same set but two operations. In an abstract sense, though there is a unique operation, no one came up with the idea. They simply thought of using usual operations. All the students used multiplication operations. For the second operation, around 40% (19 out of 47) of them used division operation, around 20% (9 of 47) of the students defined 1+1 = 1, 10% (5 out of 47) defined $a * b = a$, around 10% used multiplication in mod (2) by considering $A = \mathbb{Z}_2 - \{0\}$ and the remaining about 20% did not answer. The second question was to test whether the set {-1, 1} forms a group under two operations, namely addition and multiplication. All the students correctly identified that the set {-1, 1} is a group under multiplication and is not a group under addition with reasons. The third test item was to test whether the set of rational numbers forms a group under addition, and multiplication, and if not to suggest the way to form a group. All the students responded that the set of rational numbers is a group under addition and around 90% (42 of 47) of them answered correctly it is not a group under multiplication and 80% (38 of 47) of them suggested $\mathbb{Q} - \{0\}$ is a group under multiplication. The fourth question was to test whether $\mathbb{Z}_5$ and $\mathbb{Z}_6$ form groups under multiplication with respective modules. All answered correctly that $\mathbb{Z}_5$ and $\mathbb{Z}_6$ are not groups under multiplication. The fifth question was to test whether $_5\mathbb{Z}$ is a group under addition. All the students responded correctly. The final item was to create a new group with its operation. Around 70% (33 of 47) of the students created a new group with their operations. Also, 10% (5 of 47) of the students attempted to create a group but left the creation with a cross mark and the remaining 20% did not answer for the item.

Though there are still some challenges with very few students, the overall performance of the students in the quiz is mostly appreciable. The appreciation is not only for answering the items but also giving a variety of answers. Based on the earlier experience of the researchers, it can be said that the concept by construction is an innovative learning approach for learning mathematical concepts. Though it took a slightly longer time than stipulated in the curriculum, concept by construction is noticed to help develop foundations of axiomatic mathematical thinking. The evidence from the interview is extracted to support the long-lasting learning.

I am sure I will never forget the group that I have created and even the meaning of the group. I will start to think about myself and even encourage my friends to work as a group of students. It was a great experience when I thought of a one-way to form a group, and my friends tested the axioms and that could not satisfy all the *criteria, and I needed to think of another example. It started with the case of the set {-1, 0, 1} under addition (Interview, Member of Group A).*

The creation requires a high level of thinking and students are highly engaged in the learning process. Concept by construction is one of the scholarly contributions to help conceptual understanding, mathematical thinking, and creativity. This can be linked with creative mathematically founded reasoning, where the sequence of reasoning (new to the reasoners) is re-created, and conclusions are based on the arguments considering the components involved in the reasoning (Lithner, 2011).

From the starting part of the learning activities, it was observed that students expected everything from the teacher. Students considered themselves an empty vessel' (Jamar & Pitts, 2005) and they also expected the teacher to describe everything so that they could understand. For example, when they were asked to create a set and define an operation, they did not even try to create it for a few minutes and just waited for teachers to get complete answers from them. Once, they were made clear about the purpose and way of working, they seemed to work on examples. Initially, it was a very challenging task to create learning communities as they expected to explain everything from the teacher. Just 'NO', 'Can't' or 'No Idea', etc., were the readymade answers from the students instead of giving them a try on the task. This situation can be linked with the situation of low expectations that are crucial for learning (Jamar & Pitts, 2005). Even in the group, every member started to judge whether they knew the answer or not instead of thinking and sharing ideas within themselves.

We just used to copy the definition, examples, solutions, etc. as done by teachers on the board and understand how it was done. So, it was very difficult to think of myself and create a group. Initially, I thought, you [teacher] would explain everything as in the regular classes and we all group members waited for some time by just acting as if we had been trying. Later, when you told us that it was our task and we needed to think about it, then we started to think. When we started to create ourselves, we looked after another group member to create. I don't know about others, but I was not confident whether I could make it correctly or not. I had a fear of making mistakes. But in later days, I have developed a confidence level that making wrong is moving one step ahead in thinking. But once I *created, I needed to judge myself with the given criteria (axioms of groups). (Interview, Member of Group C).*

There was a sense of students' change in the teaching-learning approach. Students need to be motivated to learn abstract mathematics like group theory. The teacher's role is to guide in the appropriate approach to learning. Setting high expectations was one of the strategies for creating learning communities (Jamar & Pitts, 2005). Teachers need to create an open environment for thinking and considering students' mistakes as the important keys to further development in the classroom. A student's psychological readiness (Eison, 2010) to give his/her best thinking and reasoning in the learning process is most important. As a result, students developed a sense of confidence to generate new ideas and test themselves. The following portion of the interview shows how students changed their feelings.

It was the first time that I had learned something in Mathematics. I have created a group myself. It was not only me*, but all members of our group also created an example of the group. We have realized that creating a group with one or two elements is easier than creating a group having more elements. We decided to create a group individually and decided to select the best one to show you[teacher]. But later each of us realized that it was not the case of making the best, just to create and test whether it is a group or not. We have realized how difficult it is to create a non-abelian group. Also, we could not think about forming a group without using numbers. (Interview, Member of Group B).*

Students' attitudes towards abstract algebra, motivation, prior knowledge, lecturer performance, learning methods, and teaching materials are the dominant factors that cause difficulties in learning abstract algebra (Agustyaningrum et al., 2021). These interdependent factors can be addressed when we apply inquiry-based models in the undergraduate algebra classroom. The development of the sense of learning as creation is important for students as creation is the top level of cognition according to the revised Bloom's taxonomy of educational objectives. Curiosity, initiative, motivation, and readiness (Wang, 2009) are some of the attributes of being creative. The process helped students to create themselves.

Conclusion

The developed innovation plan is a reference resource for teachers who follow inquiry-based learning in their classroom. This plan has served as a best tool for building concept of group theory. The overall performance of the students in the quiz is the evidence to claim the best tool. Though it took a slightly longer time than stipulated in the curriculum. concept by construction is noticed to help develop foundations of axiomatic mathematical thinking.

Concept by construction is one of the scholarly contributions to help conceptual understanding, mathematical thinking, and creativity. From the starting part of the learning activities, it was observed that students expected everything from the teacher. The implementation of the innovative plan helped to move from low expectations that are crucial for learning to high level of engagment. This innovative plan was successful in creating an open environment for thinking and considering students' mistakes as the important keys to further development in the classroom. Students developed a sense of confidence to generate new ideas and test themselves. The innovation has developed a sense of learning as creation. The learning process helped students to create themselves and students act as mathematicians.

References

Agustyaningrum, N., Sari, R. N., Abadi, A. M., & Mahmudi, A. (2021). Dominant factors that cause students' difficulties in learning abstract algebra: A case study at a university in Indonesia. *International Journal of Instruction, 14*(1), 847−866. https://eric.ed.gov/?id=EJ1282223

Ah-Nam, L. & Osman, K. (2017). Developing 21st century skills through a constructivist-constructionist learning environment. *K−12 STEM Education, 3*(2), 205−216. http://www.k12stemeducation.in.th/journal/article/download/56/91

Arnawa, I. M., Sumarno, U., Kartasasmita, B., & Baskoro, E. (2007). Applying the APOS theory to improve students ability to prove in elementary abstract algebra. *Journal of the Indonesian Mathematical Society, 13*(1), 133−148. https://doi.org/10.22342/jims.13.1.80.133-148

Arnon, I., Cottnill, J., Dubinsky, E., Oktac, A., Fuentes, S. R., Trigueros, M., & Weller, K. (2014). *APOS theory: A framework for research and curriculum development in Mathematics Education*. Springer Nature. https://link.springer.com/book/10.1007/978-1-4614-7966-6

Braun, B., Bremser, P., Duval, A. M., Lockwood, E., & White, D. (2017). What does active learning mean for Mathematicians? *Notices of the AMS, 64*(2), 124−129. http://www.ams.org/journals/notices/201702/rnoti-p124.pdf

Caswell, C. J. & LaBrie, D. J. (2017). Inquiry based learning from the learner's point of view: A teacher candidate's success story. *Journal of Humanistic Mathematics, 7*(2), 161–186. https://scholarship.claremont.edu/cgi/viewcontent.cgi?article=1366&context=jhm

Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., John, D. S., Vidakovic, D. (1997). Constructing schema: The case of the chain rule. *Journal of Mathematical Behavior, 16*(4), 345−364. https://www.sciencedirect.com/science/article/pii/S0732312397900122

Cobb, P., Jackson, K., & Dunlap, C. (2015). Design research: An analysis and critique. In *Handbook of international research in mathematics education* (pp. 481-503). Routledge.

Doorman, M. (2019). Design and research for developing local instruction theories. *Avances de investigación en Educación Matemática, 15*, 29−42. https://aiem.es/article/download/3875/4318

Dubinsky, E. & Leron, U. (1994). Learning abstract algebra with ISETL. Springer. https://link.springer.com/book/10.1007/978-1-4612-2602-4

Eison, J. A. (2010). Using active learning instructional strategies to create excitement and enhance learning. *Jurnal Pendidikantentang Strategi Pembelajaran Aktif (Active Learning) Books, 2*(1), 1−10.

Ernst, D. C., Hitchman, T., & Hodge, A. (2017). Bringing inquiry to the first two years of college mathematics. *Problems, Resources, and Issue in Mathematics Undergraduate Studies, 27*(7), 641−645. https://doi.org/10.1080/10511970.2017.1393846

Fukawa-Connelly, T. P. (2007). *A tale of two courses: Teaching and learning undergraduate abstract algebra.* University of Maryland.

Gray, E. & Tall, D. (1993). Success and failure in mathematics: The flexible meaning of symbols as process and concept. *Mathematics Teaching, 14*, 6−10. https://wrap.warwick.ac.uk/504/1/WRAP_Tall_dot1993b-success-failure-mt.pdf

Hein, G. E. (1991). *Constructivist learnng theory*. Institute for Inquiry. http://beta.edtechpolicy.org/AAASGW/Session2/const_inquiry_paper.pdf

Henrick, E., Cobb, P. & Jackson, K. (2015). Educational design research to support system-wide instructional improvement. In Bikner-Ahsbahs, A., Knipping, C., & Presmeg, N. (Eds.), *Approaches to Qualitative Research in Mathematics Education* (pp. 497−530). Springer. https://doi.org/10.1007/978-94-017-9181-6_18

Jamar, I. & Pitts, V. R. (2005). High expectations: A "how" of achieving equitable mathematics classrooms. *Negro Educational Review, 56*, 127−134. http://edu312spring13.pbworks.com/w/file/fetch/63176576/Jamar%26PittsNER56.pdf

Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2018). Inquiry-oriented instruction: A conceptualization of the instructional principles. *Problems, Resources, and Issues in Mathematics Undergraduate Studies, 28*(1), 13−30. http://dx.doi.org/10.1080/10511970.2017.1338807

Lamichhane, B. R., & Belbase, S. (2017). Images of mathematics held by undergraduate students. *International Journal on Emerging Mathematics Education, 1*(2), 147. http://dx.doi.org/10.12928/ijeme.v1i2.6647

Lithner, J. (2011). University mathematics students' learning difficulties. *Education Inquiry, 2*(2), 289−303. https://doi.org/10.3402/edui.v2i2.21981

Melhuish, K. M. (2015). The design and validation of a group theory concept inventory (Publication No. 3722708) [Doctoral dissertation, Portland State University]. ProQuest Dissertations & Theses Global. https://www.proquest.com/dissertations-theses/design-validation

Mitchelmore, M. C. (2002, May 27–31). The role of abstraction and generalisation in the development of mathematical knowledge [Conference Presentation]. *The East Asia Regional Conference on Mathematics Education (EARCOME) (2nd) and the Southeast Asian Conference on Mathematics Education (SEACME) (9th),* Singapore. https://files.eric.ed.gov/fulltext/ED466962.pdf

Risnanosanti, Y. D. (2018). Undergraduate students' conceptual understanding on abstract algebra. *In Proceedings of the International Conference on Mathematics and Islam* (ICMIs 2018) (pp. 438−443). SciTePress. https://doi.org/10.5220/0008523304380443

Saxe, K. & Braddy, L. (2015). A common vision: For undergraduate mathematical sciences programs in 2025. *The Mathematical Association of America (MAA).* https://www.maa.org/sites/default/files/pdf/CommonVisionFinal.pdf

Wang, Y. (2009). On cognitive foundations of creativity and the cognitive process of creation. *International Journal of Cognitive Informatics and Natural Intelligence, 3*(4), 1−18. https://doi.org/10.4018/jcini.2009062301

Zaffar, A., Quraishi, S. M., & Ansari, M. K. (2013). Teaching of abstract algebra at undergraduate level. *Sindh University Journal of Education, 39*, 1−12. https://www.researchgate.net/publication/257985537