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A Numerical Perspective on Solving Non-Linear Equations: Newton Vs. Bisection

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Abstract

The purpose of this paper is to review the methods of solving non-linear equations and compare them for their procedure and suitableness for solving. The study reviewed two of methods Newton's method and Bisection method from various methods to solving nonlinear equations. The table and graphs were used to compare them and found that Newton method is measured the best due to its speedy and precise convergence to the roots. On the other hand, the Bisection method, though slower in reaching the roots compared to the Newton method, ensures convergence to the root regardless of the number of iterations.

Key words: *non-linear, root, iteration, convergence, Taylor's series.*

Introduction

The paper covers system of nonlinear equations involves multiple equations solving by two methods: Newton's method and Bisection method. A system of nonlinear equations involves multiple equations with several variables, where at least one equation is nonlinear (Iwetan et al., 2012). Solving such systems generally involves isolating a variable in one equation and substituting its value into another equation. This process is repeated until all variables are solved. Though, this approach can produce a range of different outcomes. Nonlinear algebraic equations, or polynomial equations, are characterized by polynomials set equal to zero. For example, $x^2 + x - 1 = 0$ is a polynomial equation. For and is one of the reasons algebraic geometry exists—a challenging area within modern mathematics. Even determining whether a given system of algebraic equations has complex solutions can be challenging. When graphed, nonlinear equations typically appear as curves, a single polynomial equation, root-finding algorithms can be applied to identify the solutions. However, solving systems of algebraic equations is more complex.

Nonlinear differential equations

A system of differential equations is considered nonlinear if it does not fit the criteria of a linear system. Non-linear equations in general, a problem that requires the values of the unknowns variables x_1,

$x_2,\dots, x_n$ for which $f_i(x_1,x_2,\dots,x_n) = 0$, $i = 1,2,\dots,n$
where $f_1, f_2,\dots, f_n$ are given algebraic functions of n variables. Such systems of equations arise in many areas, e.g. in numerical methods for nonlinear ordinary and partial differential equations. If $n = 1$ the single equation can be solved by a variation of effective techniques; the case of polynomial equations can give augmentation to complex solutions. The root finding problem is one of the most relevant computational problem, which appearing in science and engineering.

Newton Raphson and Bisection methods are both numerical method for determining the roots of nonlinear functions.We utilized methods like Newton's method, the Secant method, and the Bisection method to solve these equations (Paudel & Bhatta, 2023). Additionally, we explored numerical approaches such as the Runge-Kutta methods, commonly applied to solve initial-value problems for ordinary differential equations. However, these methods were specifically used to address nonlinear equations with a single variable rather than those involving multiple variables.

Numerical methods are used to provide constructive solutions to problems involving nonlinear equations. A nonlinear equation may have a single root or multiple roots. This research work will make emphasis on solving nonlinear equation in one dimension and involving one unknown $F: R \to R$ which has scalar x as solution, such that $f(x) = 0$. Numerical analysis is an extent of mathematics and computer science that creates analyses and implement algorithm for obtaining numerical solutions to problems involving continuous variables.

Materials and Methods

For the purpose of this study, the following two methods were compared; Newton method and the bisection method.

Newton Method

Newton Raphson method for calculating the root of function uses the tangent line at a point on the functions curve to approximate it. Newton's method is one of the most widely used and effective than bisection method numerical methods for solving the equation f(x)=0, where f is a function with a continuous derivative f'(Mueen, & Shiker, 2024). This method is based on the Taylor series expansion of the function f(x) around a point x_1.

$$f(x) = f(x_1) + (x - x_1)f'(x_1) + \frac{1}{2!}(x - x_1)^2 f''(x_1) \dots\dots\dots\dots\dots\dots (1)$$

Where f and its first and second derivative f' and f'' are calculated at x_1. Taking the first two terms of the Taylor's series expansion, we have

$$f(x) = f(x_1) + (x - x_1)f'(x_1) \dots\dots\dots\dots\dots\dots (2)$$

We then set (2) to zero (i.e. f (x) = 0) to find the root of the equation which yields

$$f(x_1) + (x - x_1)f'(x_1) = 0 \dots\dots\dots\dots\dots\dots (3)$$

Rearranging (3) above, we obtain the next approximation to the root, given rise to

$x = x_1 - \frac{f(x_1)}{f'(x_1)}$......................(4)

Thus generalizing (4) we obtain the Newton iterative method

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, where k ∈ N ....(5)

Let us consider an example:
$f(x) = x^3 + x - 1 = 0$
Now, apply Newton's method to the equation. Correct to five decimal places. F (0) = -1, f (1) = 1, therefore the root lies between 0 and 1. By Newton's formula

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Here, $f'(x) = 3x^2 + 1$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 + x_k - 1}{3x_k^2 + 1}$$

Now, solving the right hand side,

$$x_{k+1} = x_k - \frac{x_k^3 + x_k - 1}{3x_k^2 + 1}$$

Convergence of Newton's Method

Here in Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

This is really an iteration method where

$$x_{k+1} = \emptyset(x_k), \text{where } \emptyset(x_k) = x_k - \frac{f(x_k)}{f'(x_k)}$$

Hence the equation is
x= Ø (x), where Ø (x) = $x - \frac{f(x_k)}{f'(x_k)}$
The sequence 1, 2, 3... Converges to the exact value if | $\emptyset^{|}(x)$ | < 1
That is, if $\left|1 - \frac{[f'(x)^2 - f''(x)]}{[f'(x)]^2}\right| < 1$
That is, if $\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$
This implies that Newton method converges if $|f(x)f''(x)| < [f'(x)]^2$

Bisection Method

On bisection method we repeatedly divided an interval in half in which the roots lies. Here, assume that, we have an equation of the form f (x) = 0 whose solution is in the range (a, b) is to be determined. Also assume that f (x) is continuous and it can be algebraic or transcendental. If f (a) and f (b) are of opposite signs, then at least one root exist between a and b (Boult, & Sikorski, 1984). As a first approximation, we assume that root to be $x_0 = \frac{a+b}{2}$.

Now, find the sign of $f(x_0)$. If $f(x_0) \times f(b)$ is negative, the root lies between x_0 and b, if not it lies between a and x_0. Any one of this is true. This solution is found by repeated bisection of the interval and in each iteration picking that half which also satisfies that sign condition. The number of iteration required may be determined from the relation $\left|\frac{b-a}{2^k}\right| \leq \in$.

The general formula is; $e_k = \frac{a+b}{2}$.

Let us consider another example: Solve the equation f (x) =$x^3 + x$ - 1, correct to five decimal places. Using bisection method $f(x) = x^3 + x - 1$,correct to five decimal places.

Solution $x_k = \frac{a+b}{2}$, when k =0

A root lies between 0 and 1 such that a = 0 and b = 1

Convergence of Bisection Method

The successive approximation x_k of a root x = α of the equation f(x) = 0 is said to converge to x= α with order q ≥ 1

If $|x_{k+1} - \alpha| \leq c|x_k - \alpha|$

Here q, k >0, k and c is some constants greater than 0

When q=1 and 0<c<1, then the convergence is known as first order and c is called the rate of convergence.

Results and Discussions

Let us consider another example, that is; $f\ (x) = x^3 + x - 1$ of the nonlinear equation and illustrating the results by the two methods of solving nonlinear equations f(x) = 0. The results of the obtained by this method are presented in the given table and graphs which is given below.

Table 1. Results are given below that obtained from two methods which are discussed above:

K	Newton method $x_0=1$ x_{k+1}	e_{k+1}	Bisection method x_k	e_{k+1}
0.00	0.7500000	6.768×10^{2}	0.500000000	$1.8232x10^{-1}$
1.00	0.6860500	$3.73x10^{3}$	0.750000000	$4.596x10^{-2}$
2.00	0.6823400	$2x10^{5}$	0.625000000	$1.112x10^{-2}$
3.00	0.6823300	$1x10^{5}$	0.687500000	$2.657x10^{-3}$
4.00	0.6823278	$7.8x10^{6}$	0.656250000	$6.28773x10^{-4}$
5.00	0.6823278	$7.8x10^{6}$	0.671875000	$1.43976x10^{-4}$
6.00	0.6823278	$7.8x10^{6}$	0.679687500	$2.8418x10^{-5}$
7.00	0.6823278	$7.8x10^{6}$	0.683593750	$8.42x10^{-7}$
8.00	0.6823278	$7.8x10^{6}$	0.681640625	$5.74x10^{6}$
9.00	0.6823278	$7.8x10^{6}$	0.682617187	$7.31x10^{6}$
10.00	0.6823278	$7.8x10^{6}$	0.682128906	$7.69x10^{6}$
11.00	0.6823278		0.682373000	$7.21x10^{6}$

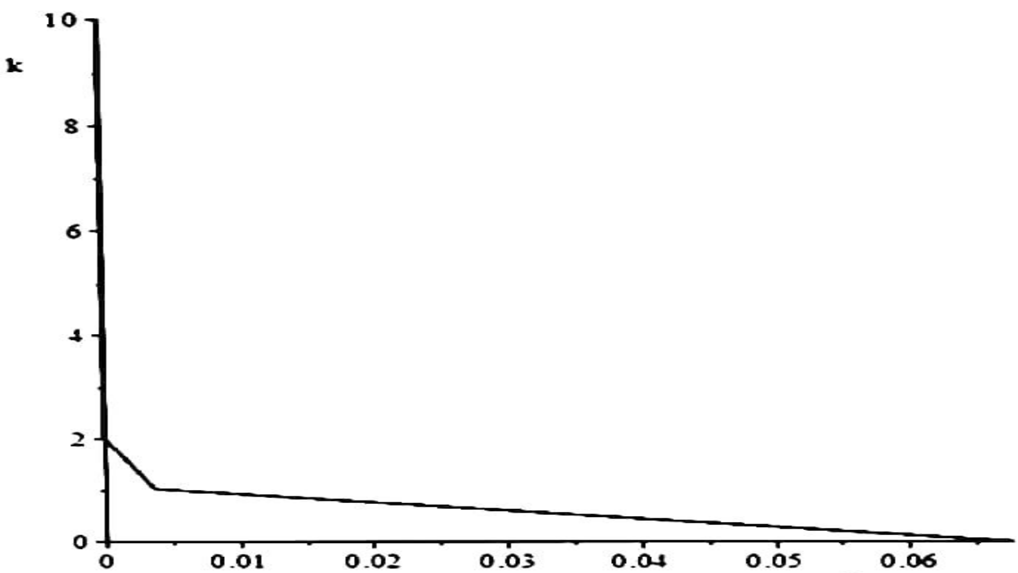

Fig1: The graph of Newton method.

The error in each k^{th} iteration in the Newton Method converges positively and remains positive with great speed and accuracy throughout the value.

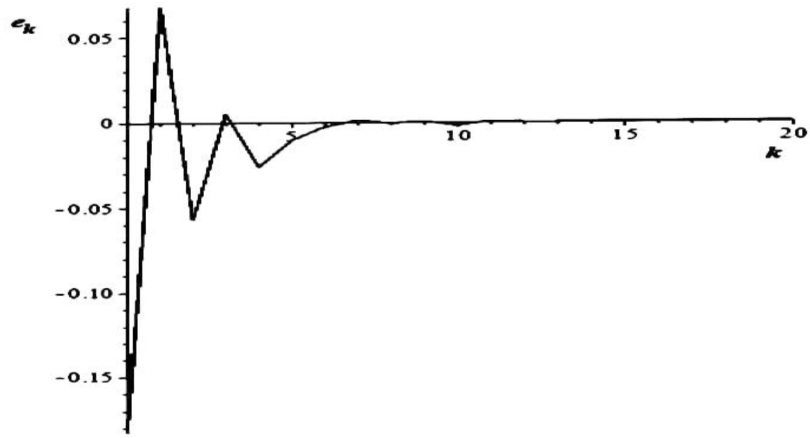

Fig2: The graph of Bisection method.

In the k-th iteration of the bisection method, the error alternates between positive and negative but becomes strictly positive at the point of convergence. The table above compares the outcomes of two methods: the Newton method and the Bisection method. The Newton method emerged as the superior approach due to its rapid and precise convergence to the roots (Srivastava & Srivastava, 2011). In contrast, the Bisection method, though slower in reaching the roots, guarantees convergence regardless of the number of iterations. Based on the methods discussed in this work, the Newton method stands out as the most effective iterative technique for solving nonlinear equations with one variable, provided the initial guess is sufficiently close to the root, as it ensures faster and more accurate convergence.

Conclusion

From the methods tested, the Newton method appeared to be the most robust and capable method of solving the nonlinear equation f (x) =0. Results obtained from the two methods above show that the Newton Method is the most efficient (fastest converges) method in finding the roots of non-linear equations seeing that it converges to the roots of the non-linear equation faster than the other three methods. That is it converges after a few iterations unlike the other three methods which converges after many iteration. This study was conduct on method of solving non-linear equation. The students can conduct such studies on other various mathematical topics like homogenous equations, second order different equations, etc.

Author Contribution

Mohan Raj Bhatt and Gita Ram Dhakal were involved in their major paperwork and observations. For the proof., Hem Lal Dhungana wrote the final drafts. All authors were involved in writing the main manuscript. All authors reviewed the manuscript.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper. The research was conducted is an unbiased and objective manner, and the authors have no personal, financial, or professional relationships but could potentially influence the results or conclusions presented in the paper.

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