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A Study of Some Well-Known Results on Fuzzy Metric Space

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Abstract

In this paper, the notation of fuzzy metric space is re-introduced with different properties and we prove some well-known results on fuzzy metric space based on George – Veeramani approach.

Introduction

To deal the vagueness and uncertainty seen in mathematics, Prof. Zadeh introduced fuzzy set and fuzzy logic in 1965. After then lots of research work have been done in the field of fuzzy logic and fuzzy sets. Fuzzy set theory is aboard view of classical set theory. Specially, fuzzy logic may be viewed as an attempt at formation of two remarkable capabilities. First the capability to converse the reason and make rational decision in an environment of vagueness , uncertainty, incompleteness of information, conflicting information, partiality of truth and partiality of possibility. Second , the capability to perform a wide variety of physical and mental task without any measurement and any computation [Zadeh 2008]. Fuzzy logic and fuzzy set theory is widely used in the field of topology. Fuzzy topology is an important and fundamental branch of fuzzy theory, with large area of research and wide range of applications. According to the demand of time, many authors have been actively participated in solving the problems in fuzzy topology to obtain an appropriate concept of fuzzy metric. Many authors have been investigated such problems in fuzzy topology from different points of view. In 1975, Karmosil and Michalek [6] introduced fuzzy metric space as the generalization of Menger Space, which is the concept of statistic metric. O. Kaleva and S. Seikkala [7] described the fuzzy metric as the distance between two points in a set as a fuzzy real numbers. In 1994, Georage and Veeramani [2] modified the concept of fuzzy metric introduced by Karmosil and Michalek. For that fuzzy metric space, they obtained Hausdroff Space. They also have shown that every metric induces a fuzzy metric. Fuzzy metric have application in making problems with uncertain and imprecise information. A. George Veeramani [ ] have defined Hausdroff topology in a fuzzy metric space. Xia, Z. Q., and Guo, F. F [11] proved that every ordinary metric space can induce a fuzzy metric space that is complete whenever the original one does. Gregori, V., Morillas, S., and Sapena, A.[3] presented examples of fuzzy metrics with respect to George and Veermani. Aage, C. T., Choudhury, B. S., & Das, K. [1] .Šostak, A.[10] presented an alternative approach to the concept of a fuzzy metric, know as revised fuzzy metric.

Definitions and Preliminaries:

Let X be universe of discourse and $A \subseteq X$. A **fuzzy set [** 12**]** in X is defined as the collection of order
pair (x , μ_A) where $\mu_A : X \to [0,1]$and $x \in X$. So that

$$A = \{(x , \mu_A(x)) : x \in X \text{ and } \mu_A : X \to [0,1]\}$$

Here, $\mu_A(x)$ is the degree of the element *x* and is called **membership function [** 12**]**. If $\mu_A(x) = 1$,
then *x* is fully included in the set *A* and if $\mu_A(x) = 0$ means *x* is not included in *A*.
The maximum value of the membership function is known as **height of the fuzzy set [** 12**].** The fuzzy
set having height one is called **normalized fuzzy set [** 12**]**. A fuzzy set *A* on X is called **convex** if for
all $x_1 , x_2 \in A$ and $\lambda \in [0, 1]$

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \text{minimum } [\mu_A(x_1) , \mu_A(x_2)]$$

Fuzzy logic[12] is defined as many valued logic form which may have truth values of variables in any real number between 0 and 1.Fuzzy logic provides a methodology for dealing with linguistic variables and describing modifiers like vary, fairly, not etc. It facilities common sense reasoning with imprecise and vague proposition dealing with natural language and serves as a basis for decision analysis and control action.

A fuzzy set ***A*** is a **fuzzy number[** 12**]** if the following conditions are satisfied

- ***A*** is a convex set
- ***A*** is a normalized set

its membership function is piecewise continuous

t – Norm [2]

A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t- norm if the following conditions are satisfied:

(i) $1 * a = a$,

(ii) $a * b = b * a$

(iii) $c * d \geq a * b$ whenever $c \geq a$ and $d \geq b$,

(iv) $(a * b) * c) = a * (b * c)$.

Distance Function[2]: Let X be a non-empty set (crisp) and d be a function from d: X ×X → R+ (non-negative real number) such that for all $x, y, z \in X$ we have

M1: $d(x, y) > 0$ and $d(x, y) = 0$ if and only if $x = y$,

M2: $d(x, y) = d(y, x)$, and

M1: $d(x, y) \leq d(x, z) + d(z, y)$.

A function d satisfying the above conditions is said to be a distance function or metric and the pair (X, d) is called a metric space.

Fuzzy Metric Space[6] (Kramosil and Michalek)

Let X be a non-empty set and $*$ is a t-norm. A fuzzy metric on the set X is a function $p: X \times X \times (0, \infty) \rightarrow [0, 1]$

(KM i) $p(x, y, t) > 0$,

(KM ii) $p(x, y, t) = 1$ for all if and only if $x = y$,

(KM iii) $p(x, y, t) = p(y, x, t)$,

(KM iv) $p(x, z, t + r) \geq p(x, y, t) * p(y, z, r)$

(KM v) $p(x, y, .) : [0, \infty) \rightarrow [0,1]$ is continuous for all $x, y, z \in X$ and $t, r > 0$.

Then the 3- tuple $(X, p, *)$ is called fuzzy metric space.

Fuzzy Metric Space [2] (George and Veeramani)

Let X be a non-empty set and $*$ is a t-norm. A fuzzy metric on the set X is a function $p: X \times X \times (0, \infty) \rightarrow [0, 1]$

(GV i) $p(x, y, t) > 0$,

(GV ii) $p(x, y, t) = 1$ for all if and only if $x = y$,

(GV iii) $p(x, y, t) = p(y, x, t)$,

(GV iv) $p(x, z, t + r) \geq p(x, y, t) * p(y, z, r)$

(GV v) $p(x, y, .) : (0, \infty) \rightarrow [0,1]$ is continuous

for all $x, y, z \in X$ and $t, r > 0$.

Then the 3- tuple $(X, p, *)$ is called fuzzy metric space.

Examples [3, 4]

1. Let X be a non-empty set and d be a metric defined on X. Suppose * is continuous

t-norm defined by $a * b = a . b$ for all $a, b \in [0, 1]$. Define a relation as $p(x, y, t) = \frac{t}{t+d(x,y)}$. Then (X , p, *) is a fuzzy metric space and p is a metric induced by the metric d.

2. Let $f: (0, \infty) \to (0, \infty)$ be an increasing function and d be metric on X. Then $p(x, y, t) = \frac{f(t)}{f(t)+ a.d(x,y)}$ for $a > 0$, is a fuzzy metric on X and (X , p, .) is fuzzy metric space.

Let (X, p, *) be a fuzzy metric space and let $x \in X$. Let $r \in (0, 1)$ and $t > 0$, then the **fuzzy**
open ball [6]with center at x and radius r is denoted by $B(x, r, t)$ and defined by

$$B(x, r, t) = \{ y \in X : p(x, y, t) > 1 - r\}$$

Let (X, p, *) be a fuzzy metric space and $\{x_n\}$ be a sequence in X. Then we say that the sequence $\{x_n\}$ **converges [9]** to x if for every $\varepsilon > 0, \exists\, N$ such that $x_n \in B(x, \varepsilon, t)\ \forall\, n \geq N$.

A sequence $\{x_n\}$ is a fuzzy metric space (X, p, *) is said to be **Cauchy** [9] if $\forall \varepsilon > 0$ and $\forall\, t > 0\ \exists\, n_o \in Z_+ : p(x_n, x_m, t) > 1 - \varepsilon\ \ \forall\, m, n \geq n_o$

Theorem:

Suppose (X, p ,*) be a fuzzy metric space and for $0 < a < 1$, the mapping defined by $p_a(x, y) = \inf\{ t : p(x, y, t) > a\}$ **is** a metric on X

Proof: Suppose (X , p, *) be a fuzzy metric space and define a relation

$$p_a(x, y) = \inf\{ t : p(x, y, t) > a\}$$

Now , we show p_a is a metric on X.

i. We have $p_a(x, y) = \inf\{ t : p(x, y, t) > a\}$.
Here $p_a(x, y)$ being the infimum of non-negative value, so that $p_a(x, y) > 0$.

ii. $p_a(x, y) = \inf\{ t : p(x, y, t) > a\}$
$\inf\{ t : p(y, x, t) > a\} = p_a(y, x)$

iii. Suppose that $x = y$ then $p(x, y, t) = 1$ and so we have
$\{ t : p(x, y, t) > a\} = (0, \infty)$
$\Longrightarrow inf\{ t : p(x, y, t) > a\} = inf(0, \infty)$
$\Longrightarrow p_a(x, y) = 0$
So that if $x = y$ then $p_a(x, y) = 0$.

The converse of the proof is omitted here.

iv. Let $x, y, z \in$ X. If $x = y$ or $y = z$ or $x = z$ then we can see that

$p_a(x,z) \leq p_a(x,y) + p_a(y,z)$. So we assume that $x \neq y \neq z$.

We have, $p_a(x,y) = \inf\{t : p(x,y,t) > a\}$

$p_a(y,z) = \inf\{t : p(y,z,t) > a\}$

$p_a(x,z) = \inf\{t : p(x,z,t) > a\}$

So that, $p(x,y, p_a(x,y)) > a$ and $p(y,z, p_a(y,z)) > a$

$\Rightarrow p(x,y, p_a(x,y)) + p(y,z, p_a(y,z)) > 2a > a$

i.e $p(x,y, p_a(x,y)) + p(y,z, p_a(y,z)) > a$

Since, p is a metric, we have

$$p(x,z,p_a(x,y) + p_a(y,z)) \geq p(x,y,p_a(x,y)) + p(y,z,p_a(y,z)) > a$$

i.e $p(x,z,p_a(x,y) + p_a(y,z)) > a$

$\Rightarrow p_a(x,y) + p_a(y,z) \in \{t : p(x,z,t) > a\}$

$\Rightarrow p_a(x,y) + p_a(y,z) \geq inf\{t : p(x,z,t) > a\}$

$\Rightarrow p_a(x,y) + p_a(y,z) \geq p_a(x,z)$

Hence, $p_a(x,y)$ is a metric on X.

Similarly , we can show that , for $0 < b < 1$, the mapping defined by

$$p_b(x,y) = \sup\ \{ t : p\ (x,y,t) > b\}$$

is also a metric on X.

Theorem [2]: Every open ball is an open set in fuzzy metric space**.**

Theorem [2]: Every fuzzy metric space is a Hausdroff space

Theorem: Let $(X, p, *)$ be a fuzzy metric space such that every Cauchy sequence in the space has a convergent subsequence, then $(X, p, *)$ is complete.

Proof: Suppose $\{x_n\}$ be a Cauchy sequence in $(X, p, *)$ and $\{x_{k_n}\}$ be a subsequence of $\{x_n\}$ such that $x_{k_n} \to x$. To complete the proof we show that $x_n \to x$. For let $t > 0$ and $0 < \varepsilon < 1$ such that

$(1 - r\ \ ) * (1 - r\ \ ) > 1 - \varepsilon$

Here, $\{x_n\}$ is a Cauchy sequence, so for $0 < r < 1$

$\forall\ t > 0\ \exists\ n_o \in Z_+ : p(x_n, x_m, t) > 1 - r \ \ \forall\ m,n \geq n_o$

Since, $x_{k_n} \to x$ so there $\exists\ k_n \in Z_+ : k_n > n_o \Rightarrow p\left(x_{k_n}, x, \frac{t}{2}\right) > 1 - r$

Now , for $n \geq n_o$, $p(x_n, x, t) = p\left(x_n, x, \frac{t}{2} + \frac{t}{2}\right)$

$\geq p\left(x_n, x_{k_n}, \frac{t}{2}\right) * p\left(x_{k_n}, x, \frac{t}{2}\right)$

$> (1 - r) * (1 - r) > 1 - \varepsilon$

Hence $p(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_o$. Hence $(X, p, *)$ is complete.

Conclusion

In this paper, we have discussed about fuzzy metric space and is re-introduced with different properties. Also we have proved some well-known results on fuzzy metric space based on George – Veeramani approach

References:

Aage, C. T., Choudhury, B. S., & Das, K. (2017). Some fixed point results in fuzzy metric spaces using a control function. *Surveys in Mathematics and its Applications,*12(2017), 23-34

George, A., & Veeramani, P. (1994). On some results in fuzzy metric spaces. *Fuzzy sets and systems*, *64*(3), 395-399.

Gregori, V., Morillas, S., & Sapena, A. (2011). Examples of fuzzy metrics and applications. *Fuzzy sets and systems*, *170*(1), 95-111.

Khalaf, A. B. ., & Abdulqader, M. W. (2021). On a fuzzy metric space and fuzzy convergence. *Proyecciones (Antofagasta, On Line)*, *40*(5), 1279-1299.

Khan, K. A. (2018). Generalized fuzzy metric spaces with an application to colour image filtering. *arXiv preprint arXiv:1809.08996*.

Kramosil, I., & Michálek, J. (1975). Fuzzy metrics and statistical metric spaces. *Kybernetika*, *11*(5), 336-344.

Kaleva, O., & Seikkala, S. (1984). On fuzzy metric spaces. *Fuzzy sets and systems*, *12*(3), 215-229.

Mohinta, S., & Samanta, T. K. (2016). Fixed fuzzy point theorems in fuzzy metric space. *Kathmandu University Journal of Science, Engineering and Technology*, *12*(2), 34 - 49.

Paudel, G. P., & Pahari, N. P. (2021). On fundamental properties in fuzzy metric space. *Academic Journal of Mathematics Education*, *4*(1), 20-25.

Paudel G. P, Pahari, N.P., Kumar, S. (2022). Generalized Form of p-Bounded Variation of Sequences of Fuzzy Real Numbers. *Pure and Applied Mathematics Journal*, *11*(3), 47-50.

Paudel, G. P., Sahani, S. K., & Pahari, N. P. (2022). Generalized Form of Difference Sequence Space of Fuzzy Real Numbers Defined by Orlicz Function. *Nepal Journal of Mathematical Sciences*, *3*(2), 31-38.

Paudel, G. P., Pahari, N. P., & Kumar, S. (2023). Topological properties of difference sequence space through Orlicz-paranorm function. *Advances and Applications in Mathematical Sciences*, *22*(8), 1689-1703.

Paudel, G. P., Pahari, N. P., & Kumar, S. (2022). Double sequence space of Fuzzy real numbers defined by Orlicz function. *The Nepali Mathematical Sciences Report*, *39*(2), 85-94.

Paudel, G. P., Rana, K. S., Shahi, N. K., & Baral, A. (2024). On Topological Properties of Fuzzy Sequence Space through Orlicz Function. *Mikailalsys Journal of Advanced Engineering International*, *1*(3), 132-144.

Paudel, G. P., Pahari, N. P., & Kumar, S. (2023). On Sequences Space of Fuzzy Real Numbers Defined by Orlicz Function and Paranorm. *Jilin Daxue Xuebao (Gongxueban) / Journal of Jilin University (Engineering and Technology Edition),* 42(02-2023), 494 - 509

Paudel, G. P., Pahari, N. P., & Kumar, S. (2022). Double sequence space of Fuzzy real numbers defined by Orlicz function. *The Nepali Mathematical Sciences Report*, *39*(2), 85-94.

Paudel, G. P., & Upadhyay, P. K. (2023).Cardiovascular Disease detection Using Machine Learning and Risk Classification Based on fuzzy model. *Hunan Daxue Xuebao/Journal of Hunan University Natural Sciences*, 60(12).

Paudel, G. P., Pahari, N. P., & Kumar, S. (2022). Application of Fuzzy Logic Through Bellmen-Zadeh Maximin Method. *Journal of Nepal Mathematical Society*, *5*(1), 41–47.

Paudel , G. P., & Rana , K. S. (2024). A Novel Topological Vector Space Framework for Early Diagnosis of Diabetic Retinopathy. *VEETHIKA-An International Interdisciplinary Research Journal*, *10*(4).

Paudel, G. P., Kumar, S., & Pahari, N. P. (2024). Fuzzy model for the validation and authenticity of avoid fraud at the time of the claim process. In *Mathematical Sciences and Applications* (pp. 45-56). CRC Press.

Paudel, G. P., Upadhyay, P. K., & Baral, A. (2023). Fuzzy Arithmetic–Based Algorithm for Identifying Medical Conditions for Better Treatment. *Mikailalsys Journal of Mathematics and Statistics*, *1*(1), 35-46.

Roopkumar, R., & Vembu, R. (2018). Some remarks on metrics induced by a fuzzy metric. *arXiv preprint arXiv:1802.03031*.

Šostak, A. (2018). George-Veeramani fuzzy metrics revised. *Axioms*, *7*(3), 60.

Xia, Z. Q., & Guo, F. F. (2004). Fuzzy metric spaces. *Journal of Applied Mathematics and Computing*, *16*(1), 371-381.

Zadeh, L. A. (1996). Fuzzy sets. *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh* 394-432.