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Exploring Undergraduate Students' Understanding of the Limit of a Function

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Abstract

The concept of limit of a function is a foundational in undergraduate Calculus and Real Analysis. While previous studies have identified various misconceptions and difficulties students face regarding this concept, there appears to be a lack of studies examining undergraduate students' understanding of the concept of limit in Nepali universities. In this regard, this study was carried out to explore undergraduate students' understanding of the concept of a limit of a function. Adopting a qualitative research design with interpretive paradigm, we conducted semi-structured interviews with six students studying Bachelor of Education (B.Ed.) in the fifth semester at one of the universities in Nepal. Four major themes emerged from the interview data through an iterative process of coding and thematizing. These themes were: (i) limited ability to construct and connect multiple representations, (ii) weak interconnections among the concepts, (iii) highly deviated subjective knowledge resulted lack of meaningful learning, and (iv) lack of conceptual understanding due to inadequate vocabulary and logical proficiency. The insights from this study have significant implications for mathematics educators and teachers suggesting instructional strategies to teach the limit of a function, ensuring conceptual understanding; for students in developing deeper understanding of the concept of the limit of a function; and for textbook authors in developing tasks, activities, and materials that support meaningful engagement with the concept of limit.

Key Words: Limit of a function, understanding, representations, subjective knowledge, Real Analysis

Introduction

The limit of a function is a foundational concept in undergraduate Calculus and Real Analysis courses as it underpins the formal definitions of other key concepts like continuity, derivative, and the Riemann integral (Bartle & Sherbert, 2005; Thomas et al. 2014). It provides students a strong foundation to understand the nature and behavior of a function nearby a given point on a line, plane, or space. Consequently, a strong grasp of the limit concepts is essential for students to be able to understand and apply these concepts in Calculus, Real Analysis, Functional Analysis, Complex Analysis, and many other areas of mathematics. Therefore, it is necessary to understand how undergraduate students make sense of the limit concepts and definitions so that teaching of such concepts can be studied and improved.

Several authors who defined and explained the concepts of limit, almost similar way besides some differences in the notations and representations. Gupta and Rani (2003) defined the concept by stating "*A function f defined on some neighbourhood of a point a, except possibly at a itself, is said to approach (or tend) to a limit l as x approaches a if and only if* $to\ each\ \varepsilon > 0\ \exists\ \delta > 0\ such\ that\ 0 < |x - a| < \delta \Longrightarrow |f(x) - l| < \varepsilon$'. Similarly, Thomas et al. (2014) introduced the concept formally and informally. Formally, they state '*Let f(x) be defined on an open interval about* x_o*, except possibly at* x_o *itself. We say that the limit of f(x) as x approaches* x_o *is the number L, and we write* $\lim_{x \to x_0} f(x) = L$*, if for every number* $\varepsilon > 0$*, there exists a corresponding number* $\delta > 0$*, such that for all x,* $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$" (p. 92). Informally, the authors defined the limit of a function as "*Let f(x) be defined on an open interval about* x_o*, except possibly at* x_o *itself. If f(x) gets arbitrarily close to L for all x sufficiently close to* x_o*, we say that the limit of f(x) as x approaches* x_o *is the number L, and we write* $\lim_{x \to x_0} f(x) = L$" (p. 77).

Definitions in both of the books are similar except for some variations in notations. These definitions have been adopted in most of the textbooks and class notes, such as Lazarus (2017), Pilachowski (n.d.), and Zeff (2023), to list a few. The reason behind mentioning the formal definition here is that the extent of learning of the concept of limit of function depends upon the understanding the meaning of formal definition.

Regarding mathematical understanding, Hiebert and Carpenter (1992) mention that "mathematics is understood if its mental representation is a part of the network of representations; the degree of understanding is determined by the number and strength of its connections" (p. 67). Hiebert and Carpenter further state that "mathematical idea, procedure, and fact can be understood thoroughly if it is linked to existing networks with stronger or more numerous connections" (p. 68). They focused on the representations and the connection between associated representations that existed in the mind (network of thoughts) of the learners. Similarly, Pritchard (2009) states that adding something to a mental schema (mental image of objects or ideas) and connecting it to other items indicates understanding and it is gained more by constructing more connections between schemas. Pritchard (2009) further highlighted the role of constructing mental presentations and connecting them with existing cognitive structures. Based on these ideas, understanding of mathematical concepts depends upon the existing network of representations of a learner and his/her ability to link new concepts with this network. In this research, the extent of undergraduate students' understanding of the limit of a function depends upon the strength of the connection of its representations with representations of other associated concepts and the strength of its link with existing prerequisite cognitive structures such as schema of neighborhood, absolute value, biconditional statement, etc. To best describe the understanding of mathematical concepts, it is necessary to discuss the idea of representations.

Describing the role of representation in mathematical understanding, Hiebert and Carpenter (1992) state that by analyzing the representations constructed for a given concept by learners, it is possible to determine their understanding of that concept. They mention two types of representations: internal representations (concerning mental imagery) and external representations (concerning spoken language, pictures, drawings, and symbols). However, determining internal representations directly is not an easy task, therefore we try to explore students' external representations to understand their internal representations. Mainali (2021) mentions four modes of external representations: graphic, numeric, algebraic, and verbal. He mentions figures, diagrams, pictures, and drawings as graphic representations; ordered lists and tables as numeric representations; symbols and formulae as algebraic representations; and expression using language form as verbal representation. For a better understanding of mathematical concepts, a learner needs to be able to construct these multiple representations and explore the relationship between such representations. For example, if a student can interpret the symbolic form of the definition of limit, interprets its geometrical meaning, explains it verbally, gives numeric examples of limit of a function at a point, and can interpret the relationship between these four representations, then we can infer that he/she has good understanding of the limit of a function.

Several studies have been conducted on the students' performance in terms of the limit of a function. Bezuidenhout (2001) mentions that a maximum of the six hundred thirty students of different universities in South Africa possessed isolated and procedural knowledge of limit rather than its conceptual understanding. This result indicates that there was a lack of connection between associated concepts. Concerning representations, Hrnjicic et al. (2022) mention that transitioning from an algebraic to a verbal representation of a function concept was very problematic for their participants. This result shows that there was lack of ability to translate among multiple representations. Further, Adiredja (2014) states that the order of epsilon (ε) and delta (δ) created an obstacle to understand the formal definition of the limit of a function. This means that the epsilon-delta ($\varepsilon - \delta$) relation is one of the sources of difficulty behind the weak understanding of the limit of a function. Further, Oktaviyanthi et al. (2018) reported that students could not give appropriate meaning to one or more of the five phrases 'for every epsilon, there is a positive delta, in such a way that for every x, if $0<|x-a|<\delta$, then $|f(x)-L|<\varepsilon$' so that they could not interpret the limit of a function meaningfully.

Further, Adhikari (2020) found that students had a lack of understanding of concept of the ε- δ definition of the limit of a function. These students had difficulty in stating ε- δ definition because they were not taught the meaning of this definition but they were taught to find limiting value using certain rules. These results show that there is a pedagogical reason for students' limited or no conceptual understanding of the limit of a function at a point in Nepali higher educational institutions. Such issue has been reported in other studies too. For example, Munyaruhengeri et al. (2022) concluded that most (235 of out 252) participants felt difficulty in determining limits using definitions and the reason behind this situation is due to teachers' teaching approach that focuses only on the procedural aspects neglecting conceptual understanding. It indicates that students did not have proper conceptual knowledge of the definition of limit due to pedagogical limitations. Therefore, they were not able to apply the concept in determining limit. Likewise, Denbel (2014) and Messies (2024) reported misconceptions such as limiting value as a functional value, a function that is undefined at a certain point does not have a limit at that point, the limit of a function can be found by substitution method, and when the function has a limit at a particular point then that function is continuous at that point.

Several past studies mentioned misconceptions and difficulties associated with the limit of a function (e.g., Baptiste, 2024; Bokhari & Yushau, 2006; Denbel, 2014; Juter, 2003 & 2005a, b; Karatas, et al., 2011; Liang, 2016; Lynch, 2000; Maharaj, 2010; Messias, 2024; Szydlik, 2000; Wakhata et al., 2023). Despite several studies on limit of a function, there are not adequate studies on how students make sense of the limit concepts and how they understand the meaning of limit of a function at a point on a line or plane or space. The first author also observed, during the classroom discussion and in the examinations of a course in Real Analysis, weak performance of undergraduate students in the limit of a function and other related concepts in Real Analysis. Thus, the problem of understanding the limit of a function has been observed at local and global context. The first step in improving students' understanding of the concept of limit of a function is to know the status of current understanding of students' knowledge of limit of a function. Also, there is no information in our knowledge about any past studies on undergraduate students' understanding of the concept of the limit of a function. In this regard, this study aimed to fill the gap in the literature. Therefore, the objective of this study was to explore undergraduate students' understanding of the concept of a limit of a function. The research question for the study was: How do Nepali undergraduate students understand the concept of limit of a function?

Methods

This study adopted a qualitative research within an interpretive paradigm. The research participants were six out of nine undergraduate students, selected purposively, from the B. Ed. fifth semester in a public university in the Western region of Nepal. We selected these participants based on their voluntary participation, informed consent, and enrollment in Bachelor of Education with a major in mathematics. The first author contacted the potential participants and recruited them for the interviews. We prepared a interview guideline consisting of questions from the limit of a function and one-sided limits of functions, focusing on the definitions given by Gupta and Rani (2003) and others. The interview guideline was supplemented by tasks for students to explain their understanding of limit of a function and related concepts, represent them, and elaborate them. Hence, all the interviews were task-based. The major focus of the interview schedule was on exploring the understanding of the concept of a limit and the one-sided limits of a function. We interviewed all students face-to-face, and the average time of the interviews was about one hour.

The interview data were first transcribed verbatim. Then, the interview transcripts were read thoroughly for a general understanding of the data. The data were coded with primary codes as a basis to categorize students' thinking and understanding of limit of a function and related concepts. These primary codes were grouped into thematic structures for a broader understanding of their meaning, representations, and expositions of limit of a function. We noticed themes related to interpretations, illustrations, concept images, representations, and understanding during the interview. For the interpretation, we used the idea of subjective mathematical and objective mathematical knowledge by Paul Ernest (1991), the concept of understanding mathematical concepts by Hiebert and Carpenter (1992), and the idea of representations by Mainali (2021).

To ensure the authenticity and integrity of the data and findings, we prioritized a faithful representation of students' perspectives through their own narratives, thereby addressing the credibility of the study. Particular attention was given to accurately conveying students' voices and experiences as they engaged with the concept of the limit of a function. To enhance truthfulness and confirm the alignment between participants' intended meanings and the transcribed data, we employed member checking. Participants were given the opportunity to review and verify their transcribed responses, allowing them to identify and correct any discrepancies between the transcript and their actual views on the concept of a limit.

The four themes identified in the analysis were derived from the authentic difficulties and challenges students encountered during task-based interviews. These themes also align with commonly reported issues experienced by mathematics educators, supporting the transferability of the findings to broader instructional contexts. This alignment underscores the potential relevance and applicability of the results beyond the immediate study group, particularly for informing pedagogical practices in undergraduate mathematics education.

Results and Discussion

The iterative process of data analysis by coding and thematizing provided four central themes related to construction, connection, representation, and meaning of limit of a function. We interpreted these themes based on the three theoretical perspectives: subjective and objective mathematical knowledge (Ernest, 1991), concept of understanding (Hiebert & Carpenter), and representations (Mainali, 2021).

Limited Ability to Construct and Connect Multiple Representations

We observed two things associated with multiple representations of concepts. First, weak performance in connecting several representations and the second is weak performance in exploring relationship among such representations. We found that students could not represent several concepts associated with limit of a function in a real number line. For example, to represent $0 < |x-2| < 1$ in a real number line, Student 1 drew a number line and marked numbers on it but incorrectly shaded the region from -2 to 2 for the above inequality. To represent limit on a plane, he drew lines $x = a - \delta$, $x = a + \delta$ before the lines $y = l - \varepsilon$ and y= $l + \varepsilon$. This indicates that he memorized graphical representation. Similarly, Student 2 marked $a - \delta \ and \ a + \delta$ on a number line but was unable to show the position of x on it. Likewise, Student 3 was also unable to represent $|f(x) - l| < \varepsilon$ in a number line. Student 5 mistakenly marked $a + \delta$ left to $a - \delta$, and $l + \varepsilon$ left to $l - \varepsilon$ in a real number line. In this way the participants did several mistakes while representing $\varepsilon \ and \ \delta$ intervals on a number lines. Further, students failed to represent concepts on a plane (e.g., X-Y coordinate plane). For example, none of the participants got success in sketching the graphical representation of limit, left hand limit and right-hand limit of a function. Likewise, none of the six participants could sketch the graph of $y = x^2$. Thus, students performed weakly in constructing graphical representation of limit and one-sided limits. Similar to this result, Hashemi et al. (2014) mentioned that the participants had difficulty constructing graphical representations of calculus concepts.

Similarly, students could not interpret the symbolic form of the definition of limit of a function in an acceptable form. This indicates weak performance in constructing verbal representation. There was not sufficient discussion on the numeric representation during the interview. As none of the participants considered the numeric representation into consideration during interview. It can be argued that students did not have numeric representation in their mind. However, we found that students were able to recall, partially, an algebraic representation of the definition of limit of a function and one sided limits. We observed the dominance of algebraic representation as the first concept image. For example, Student 1 responded the question 'what came in your mind for the limit of the function?' by saying, *"the limit symbol* $\lim_{x \to a} f(x) = l$ *and its* $\varepsilon - \delta$ *condition came in my mind*". Similarly, Student 2 said, "*the symbols* $x \in (a, a + \delta)$ *and* $\lim_{x \to a^+} f(x) = l$ *came in my mind*", in response of the question: What came in your mind when I said one-sided limits?

Similarly, other participants also responded that the symbols came in their mind for the limit and one-sided limits. These results align with the finding in Anna and Dmitry (2011), which states that participants relate concepts with associated mathematical symbols. It also aligns with the finding of Habre and Abboud (2005) that, in a calculus class, thinking of the most students was dominated by algebraic representation. One of the reasons behind this might be the teaching approach that used to start the concept of limit of function from symbolic definition. Tall and Vinner (1981) found that the first concept image of their participants were geometrical picture of a tangent because limit concept starts from differentiation in their course.

The performance of students in making connections between different representations was weak. Student 1 and Student 2 memorized algebraic representations and graphical representations (partially) of definition of limit of a function at a point in an isolated way without connection between graphical representation and algebraic representation. Student 3 had memorized interval notation and absolute value notation in an isolated way without having an understanding of their relationship. Other participants too could not make any connection between representations of simple concepts like a neighborhood. Student 5 could not write verbal definition in symbolic form and said, "*I could not understand the figure*". Similarly, Student 6 memorized isolated representations of absolute value, but in the case of limit and continuity, he was unable to demonstrate any representations other than symbolic form. Thus, the participating students either possess only algebraic representation or memorized multiple representations in isolated forms, but were unable to establish appropriate connections among multiple representations of the limit of a function. One of the reasons behind this might be that procedurally teaching these students to reproduce mathematical definitions through rote memorization (Luitel & Taylor, 2005) rather than engaging them in the construction of multiple representations. These findings align with the finding of Hrnjicic et al. (2022), who mention that participants were very less successful in transitioning from algebraic to other forms of representations. This view supports our finding that students become unsuccessful in connecting algebraic representations with other representations. As Adu-Gyamfi (2003) found difficulty in understanding a concept was caused by the inability to see relationships among several representations, the participants in the current study might have felt difficulty in understanding of limit of a function due to lack of connections across different meanings and representations. These results also indicate that the participants of this study could have less chance of being proficient in the concept of limit of a function that requires the ability to translate among multiple representations (Mainali, 2021).

Weak Interconnections among the Concepts

We observed two weaknesses concerning the skill of connecting networks of associated concepts. First, the participating students failed to connect different parts of definitions so that they could not generate holistic meaning of the definitions. Second, those students could not explore the relationship between associated concepts such as left hand limit (LHL), right hand limit (RHL), and the limit of a function.

For example, Student 1 interpreted the definition of limit by writing "$if\ \forall\varepsilon > 0, \exists\ \delta > 0: 0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$", which is incomplete. In addition, he had misinterpreted that, "*if f is defined on the nbd of 'a' then limit of f(x) is l*". It seems that Student 1 was unable to connect parts of a definition to make holistic meaning. Similarly, regarding the relation of epsilon and delta, Student 2 said, "*delta part is given and epsilon part is to be proved*". He ignored several parts of a definition while interpreting the definition of limit of a function. Likewise, Student 3 got confused in identifying two statements of definition in biconditional form and also was unable to explain epsilon delta relation. Similarly, Student 6 interpreted the definition by saying, "$\varepsilon > 0, \exists\ \delta > 0;\ \varepsilon\ is\ greater\ than\ \delta;\ \exists\ \delta > 0$, such that $0 < x - a, f(x) - l < \varepsilon$" which is simply a collection of isolated facts and he could not see the relation between these parts. In this regard, Adiredja (2014) mentioned that the temporal order of delta and epsilon is obstacle for students in understanding the structure of the formal definition, which aligns with our current finding. These results indicated that the participants possessed some disconnected networks of fragmented concepts in the definition which created a barrier for proper understanding (Hiebert & Carpenter, 1992). Because of lack of ability to interconnect different parts of the definition, the subjective knowledge (Ernest, 1991) of students seemed deviated from the expected goal of defining limit of a function at a point.

Moreover, none of the students except Student 1 were able to state the relationship between the left hand limit (LHL), right hand limit (RHL) and the limit of a function at a point. Student 1 demonstrated through images of the concept map in such a way that the neighborhood is at the base for limit, the limit depends upon the left-hand limit (LHL) and right-hand limit (RHL), and continuity also depends upon LHL and RHL. However, he was unable to explain how the limit of a function depends upon LHL and RHL. Furthermore, participants understood the limit of a function in isolated ways and were unable to state its relation with continuity and derivatives. In terms of Hiebert and Carpenter (1992), their learning seemed weak in making connection among these concepts leading to a lack of understanding the concept of limit of a function at a point. These results align with the findings of Susilo and Darhim (2019) who mentioned that only 6.63% of the participants were able to explain the relationship between limit and continuity of function.

Highly Deviated Subjective Knowledge Resulted Lack of Meaningful Learning

We noticed several misconceptions concerning one sided limits and limit of a function. One of the misconceptions was interpreting the limiting value as the functional value at a. For example, Student 1 said, "*when 2 is replaced for x, the value is 4*", when we asked him to interpret $\lim_{x\to 2}\frac{x^2-4}{x-2}$ = 4. Similarly, student 2, Student 3 and Student 4 interpreted $\lim_{x\to a} f(x) = l$ *as* "*if we put a in place of x, we get the value l*", "*when a is replaced for x in f(x) the value l is obtained*" and "*If value of x is near to a then value is l*", respectively. Two other students also interpreted limit as a functional value. Thus, all the participants had misconception of limiting value as a functional value. Similar types of misconceptions were found in other studies too. For example, Denbel (2014) reported that the participants had misconception that limit could be found by substituting the point in a function. Likewise, Messies (2024) mentioned that the participants possessed misconception of limiting value as a functional value. Further, position of x in $x \to a, x \to a^+,$ *and* $x \to a^-$ is also misunderstood by students. For example, Student 1 said, "*in* $x \to 2$*, x lies exactly at 2*". Similarly Student 2 said, "$x \to a^+$ *is x tends to a from right and it means x tends to +a*". During the discussion on $\lim_{x\to 3}\frac{x^2-9}{x-3} = 6$, Student 3 said, "*x approaches 3 means x takes values less than 3*" and student 4 said "*x tends to 3 means x=3*". Regarding these symbols, Student 5 interpreted, "*x tends to a from right means x is less than a*" and "*x tends to a from left means x is greater than a"*. Student 6 interpreted $x \to a$ as, "*x implies a.*". He further interpreted that x is positive in RHL and x is negative in LHL. Regarding LHL, he said, "x tends a^- means x goes towards –a" and regarding RHL he said that, "*x tends a^+ means x goes towards +a*". Thus, students demonstrated several misconceptions regarding the symbol of 'tends to', LHL, RHL, and the limit of a function at a point. In this case also, the subjective knowledge of participant on the symbol '*tends to*' was highly deviated from corresponding objective knowledge (Ernest, 1991).

Furthermore, we observed that students failed to explain appropriate meaning to the terms, symbols, and statements. For example, Student 1 interpreted the expression 0<|x-a|<δ as "*modulus of x-a is less than δ and x lies between a- δ and a+ δ*". It indicates that he ignored the role of 0<|x-a|. The other participants performed weakly in generating meaning compared to Student 1. Student 2 did not perform any noticeable meaningful learning. Similarly, when we asked them to interpret the definition of a limit, Student 5 said, "*one function…., any point,… from one side…*" which is meaningless. In most of the cases other participants were unable to give any specific meaning of limit of a function at a point. Therefore, there was not any noticeable evidence regarding meaningful learning of the definition of a limit of a function. Regarding learning of formal definition of a limit of a function, Oktaviyanthi et al. (2018) mentioned that at least five phrases 'for every epsilon, there is a positive delta, in such a way that for every x, if $0<|x-a|<\delta$, $|f(x)-L|<\varepsilon$' need to be understood. Nonetheless, their participants were unable to give appropriate meaning to these phrases. In our study also, students were unable to give meaning to such phrases. The situation implies that, there was very less chance of being meaningful learning because for meaningful learning the learner needs to generate appropriate meaning by integrating new information into existing schema (Novak, 2002). The results showed that existing cognitive structures of our participants seemed below the minimum requirements. Since all the participants possessed several misconceptions and their subjective knowledge was deviated too far from the objective knowledge, there might be a problem in the pedagogical approach followed by the teacher. Hence, time has come to reflect on the pedagogy and content as suggested by Pant (2017) to deconstruct and reconstruct the pedagogical approach. However, how to deconstruct and reconstruct these pedagogical actions for meaningful learning of limit concept of a function needs further attention by the future researchers with large scale data through participation of more students and teachers in such studies.

Lack of Conceptual Understanding due to Inadequate Vocabulary and Logical Proficiency

Mathematical language in which mathematical expressions are to be studied and the logical aspects of the mathematical expressions are equally important to the mathematical content itself. However, we found that students did not possess sufficient knowledge and skills in mathematical language and logic. Although the two aspects are closely related, we discussed these in separate paragraphs.

When we asked Student 2 to explain the definition of a limit, he said, "*Possibly there is a point, it does except, and x does approach, there is point a on line and it averages in l*". This is a meaningless expression and it indicates weak mathematical language to express meanings and ideas of limit of a function. Not only Student 2 but all the students could not say the appropriate meaning of the phrase '*except possibly at a itself*', which is also an indication of a lack of knowledge of the mathematical language. Further, we asked Student 5 to explain the meaning of '*x tends to a*' and he replied, "*There is a value 'a' in x*", which is also a meaningless explanation. Expressing her view Student 4 said, "*I do not know the exact meaning of the definition because of the (English) language. I feel difficulty in connecting different parts of the definition because of the difficulty in understanding language*".

Thus, we found that students faced difficulty in generating the meaning of phrases and definitions written in English; this, in turn, created a barrier to meaningful learning. One of the reasons behind such a problem might be that the participants could not get the opportunity to study English language at their school level. One of the studies conducted in Nepal by Chand (2021) reported that because of the lack of ability to give meaning to lengthy sentences written in English, students fail to understand the holistic meaning of the definition. Difficulty in understanding the mathematical content, particularly definitions and proofs, is not limited to Nepal. Moore (1994) stated that inadequate knowledge of the language is the source of difficulty in proof construction. Similarly, Raupu et al. (2020) mentioned that their participants made reading errors in the calculus course. Thus, we found lack of knowledge of the English language created a barrier to meaningful learning (Novak, 2002) because it is almost impossible to generate the meaning of a sentence without knowing the meaning of different terms, phrases, and symbols included in the sentence.

Regarding the logical aspect, we observed weaknesses in dealing with quantifiers, negating definitions, and grasping the logical structure of the definitions. All the students were unable to understand the role of quantifiers used in the definition of a limit. For example, we discussed on the $\varepsilon - \delta$ part '$\forall\varepsilon > 0 \; \exists \; \delta > 0{:}\, 0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$' of the definition. Student 1 said that, "*the condition must be satisfied for one* ε *because* $\forall$ *means to each and each means one*". The participant did not have clear concept of '*to each*'. Similarly, Student 4 said, "*there exist means for some and it means for maximum*". She got confused in the meaning of the quantifier '*there exist*'. Likewise, student 5 said that, "*epsilon must be positive and one delta must exist*" which clearly indicates the lack of understanding of role of quantifiers. Other participants also had lack of ideas related to the role of quantifiers.

Further, we noticed the weakness in forming negation of conditional statements associated with the definition. We asked the participants to write the condition for $\lim_{x\to a} f(x) \neq l$. Responding it, Student 2 wrote, "*If* $\lim_{x\to a} f(x) \neq l$ *then* $\exists\varepsilon > 0 \; \forall\delta > 0{:}\, 0 < |x-a| < \delta \Rightarrow |f(x) - l| \leq \varepsilon$". Here, student 2 did not know the hidden meaning of $\forall x$, misunderstood that negation of $<$ is $\leq$, and made a mistake that negation of conditional statement is again a conditional statement. Similarly, Student 4 wrote that, "$\lim_{x\to a} f(x) \neq l$ *iff* $\exists\varepsilon > 0\forall\delta > 0{:}\, 0 < |x-a| < \delta \Rightarrow |f(x) - l| \leq \varepsilon$". Her mistakes were similar to that of Student 2 except the biconditional structure. She added, "*negation of p implies q is q implies p*" which indicates that she did not know the rule concerning negation of statement. Similarly other participants also were unable to write negation of the $\varepsilon - \delta$ condition. It indicates that students did not have knowledge of the condition for a particular point to be not a limit of a given function. Moore (1994) mentioned that a lack of logical knowledge is one of the sources of difficulty in proving a theorem. The current study added what kind of difficulties students face during understanding definitions. The abovementioned findings also align with the results of the finding of Widiati and Sthephani (2018) who states their participants faced the problem of thinking logically. Moreover, Ernest (1991) mentions that logical knowledge is the foundation for grasping mathematical knowledge but we found that the participants demonstrated a weak performance in logical aspects of limit of a function.

Implications

The findings of this study highlight substantial gaps in undergraduate students' understanding of the concept of the limit of a function at a point. Specifically, students exhibited limited proficiency in constructing and representing this foundational idea, as well as in coordinating the various representations associated with it. Their difficulties extended to translating between graphical, numerical, verbal, and symbolic forms, suggesting an underdeveloped ability to synthesize and internalize the concept holistically. Moreover, a marked divergence was identified between students' intuitive or subjective interpretations of the limit and its formal mathematical definition.

These limitations appear to be closely linked to inadequacies in students' mathematical language and logical reasoning skills. Their restricted ability to articulate, represent, and interrelate mathematical ideas underscores the need for a pedagogical shift. This presents an important opportunity for mathematics educators—particularly those teaching undergraduate calculus and analysis—to critically reassess existing instructional practices. Central to this reassessment should be an emphasis on fostering students' deep conceptual understanding, promoting meaning-making processes, and encouraging the use of diverse and interconnected representations. The four thematic outcomes of the study offer concrete and actionable guidance for curriculum development and instructional design. To bridge the gap between abstract mathematical theory and students' conceptual understanding, the following strategies are recommended:

- Intentional integration of multiple representations, including graphical, symbolic, numerical, and verbal forms.
- Development of robust connections among these representations to reinforce conceptual coherence.
- Emphasis on precise and expressive mathematical language, facilitating logical reasoning and clear communication.

Textbook authors and curriculum designers, in particular, are encouraged to incorporate these principles in the creation of learning materials. This includes enhancing the clarity and depth of conceptual explanations, designing tasks that promote representational fluency, and guiding students in interpreting abstract ideas—not only for the concept of the limit, but across other key mathematical topics as well.

Conclusion

The study revealed four key concerns in teaching and learning of a key foundational concept, the limit of a function, in Calculus and Analysis. These concerns were related to construction, connection, deviation, and elaboration (with language and logic) of meaning of limit of a function. Students might have difficulty in constructing graphical and verbal representations of limit of a function, which in turn creates further difficulty in understanding the limit and translating multiple representations. To address such difficulties, teachers might engage students in constructing multiple representations and exploring how to connect such representations. Similarly, undergraduate students might memorize or learn concepts in isolated ways rather than understanding by connecting them with other associated concepts. To make learning of abstract concept such as limit of a function richer and more meaningful, teachers might engage students in exploring the relationship of a particular concept with other associated concepts. Likewise, students might generate meanings for terms, symbols, phrases, and definitions that are highly deviated from the acceptable meanings. Moreover, students might not understand concepts like limit and one-sided limits because of a lack of knowledge and skills associated with language and logic. Therefore, teachers might help students with logical aspects and language aspects to support meaningful learning. By studying the difficulties, misconceptions, and understanding of undergraduate students regarding the limit of a function, the teachers might improve their teaching approach that can supports conceptual understanding of the abstract concept, such as limit of a function. Researchers might conduct a study exploring pedagogical approaches that support the conceptual understanding of abstract mathematical concepts such as limit, derivative, integration, differential equation, and beyond by undergraduate and graduate students in higher education institutions.

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