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Importance of Definitions in the Teaching and Learning of Mathematics: Examining the Case of Linear Algebra

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Abstract

This study explores the role of definitions in teaching and learning linear algebra, emphasizing their impact on understanding and problem-solving. By analyzing existing research, the paper highlights how precise mathematical definitions aid conceptual clarity while reducing misconceptions. It highlights the importance of integrating extracted and stipulated definitions into teaching to improve student engagement and learning outcomes. Recommendations are provided to help educators enhance pedagogical practices in linear algebra, fostering deeper comprehension and application of abstract mathematical concepts.

Keywords: Concept image, extracted definations, linear algebra, stipulated definations.

Introduction

Mathematics is a language of precision, where definitions serve as the foundational building blocks for understanding abstract concepts and developing logical reasoning. In the context of mathematics education, particularly in linear algebra, definitions play an important role in fostering conceptual clarity and problem-solving skills. However, students often struggle to grasp the formal nature of mathematical definitions, leading to misconceptions and fragmented understanding of key concepts (Edwards & Ward, 2008; Vinner, 1991).

Research indicates that while students may develop partial or intuitive understandings, their concept images often conflict with formal definitions, hindering their ability to connect foundational ideas with advanced topics (Tall & Vinner, 1981). Linear algebra, known for its abstraction, poses unique challenges, as students must navigate a dense network of definitions related to vectors, matrices, and spaces (Berman & Shvartsman, 2016; Carlson, 1993). Despite these challenges, the importance of explicit instruction in mathematical definitions remains underexplored in practical teaching contexts.

This paper aims to address this gap by critically examining the role of definitions in the teaching and learning of linear algebra. Drawing from existing literature, it evaluates how extracted and stipulated definitions can shape student understanding and highlights strategies to bridge the gap between informal concept images and formal mathematical frameworks. By exploring these issues, the study seeks to provide actionable insights for educators, ultimately enhancing teaching practices and fostering deeper comprehension of linear algebra.

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Literature Review

The importance of definitions in mathematics education has found its way through many research papers. Edwards and Ward (2008) point out that for mathematicians, as well as for students, definitions are a critical tool for proper communication among them. This is because it serves to be the very base upon which logical reasoning and writing of proofs are dependent in higher mathematics. Vinner (1991) describes the difference between everyday language definitions and mathematical definitions and stresses that it is left for students to come to terms with the definitions they are required to understand and not to be misled by possible misconceptions.

The research findings offered by Berman and Shvartsman (2016) support the significance of formal definitions in linear algebra, saying that students find abstract material problematic since they do not know the definitions. They conducted an experiment where the involvement and understanding of the students were significantly improved by adding theoretical questions related to definitions into the assignment bank. In support, Dorier et al. (2000) and Carlson (1993) outline the existing difficulties that students experience in learning linear algebra. They attribute this to be an effect of the lack of teaching and learning time being put on formal definitions back in high schools.

Tall and Vinner (1981) introduced a new term, "concept image," which refers to the whole mental representations attributed to mathematical concepts. Work of this sort suggests that students frequently possess only partial or incorrect concept images and makes clear how such can get in the way of understanding formal definitions. One group of researchers suggested a solution to this: mathematics education should pay greater attention to the unique characteristics of mathematical definitions. In sum, work in this area argues for an educational turn that would emphasize definitions in teaching mathematics and learning, especially in the case of areas very abstract, like linear algebra.

Methodology

This study adopts a textual analysis approach to explore the role of definitions in the teaching and learning of linear algebra. Textual analysis provides a systematic way to investigate theoretical insights and practical implications drawn from the existing literature on mathematics education. This method was selected for its ability to present diverse perspectives and identify recurring themes.

Data sources

The research draws on a variety of academic sources, including:

i.) Peer-reviewed journals focused on mathematics education and pedagogy.
ii.) Books and research articles discussing the theoretical framework and practical challenges associated with teaching definitions.
iii.) Studies addressing the specific difficulties encountered by students in linear algebra, particularly regarding their understanding of definitions and conceptual frameworks.

Theoretical frameworks

The analysis is guided by three educational theories:

i.) Constructivism: Interprets how students build their understanding of linear algebra concepts through engagement with definitions.

ii.) Cognitive Load Theory: Examines the cognitive challenges students face when learning mathematical definitions.
iii.) Concept Image Theory: Explores the alignment or misalignment between students' mental representations and formal definitions.

These frameworks provide a structured approach to understanding the educational impact of definitions in mathematics.

Analysis process

i.) Text Selection: Literature was carefully selected based on its relevance to the research topic, focusing on works that specifically address definitions in mathematics education.
ii.) Thematic Categorization: Key insights were paraphrased, categorized, and organized into themes such as the distinction between stipulated and extracted definitions, the role of concept images, and strategies to address misconceptions.
iii.) Synthesis: The findings were synthesized to derive actionable insights and recommendations for improving teaching practices in linear algebra.

Validation

Reliability of findings was ensured by cross-referencing themes across multiple sources and analyzing insights to ensure consistency.

Results and Discussion

Understanding definitions

Definitions play a key role in mathematics, but their creation and use differ from those of "everyday language" definitions (Edwards & Ward, 2008).

Philosopher Richard Robinson (1962) and lexicographer Sidney Landau (2001), in their work classified definitions into two types namely: extracted definitions and stipulated definitions.

- Extracted definitions: That sort of word-thing definition in which we are explaining the actual way in which some actual word has been used by some actual person (Robinson, 1962, p. 135). Definitions that are based on examples of actual usage, definitions extracted from a body of evidence (Landau, 2001, p. 165).
- Stipulated definitions: Explicit and self-conscious setting up of the meaning-relation between some word and some object, the act of assigning an object to a name (or a name to an object) (Robinson, 1962, p. 59). Landau (2001, p. 65) says such definitions "are imposed on the basis of expert advice" with the goal of "ease and accuracy of communication between those versed in the language of science."

In simpler terms: extracted definitions are like dictionary definitions. They explain how a word is commonly used by people. For example, the word "circle" is commonly understood to mean a round shape where every point is the same distance from the center. On the other hand, stipulated definitions are specific definitions created for clarity and precision. They define a term in a very exact way to help with understanding complex ideas. For example, in geometry, a "circle" might be specifically defined as "the set of all points in a plane that are at a fixed distance from a given point". Similarly, everyday understanding of probability might include notions of chance or luck, while in mathematics; probability is defined as the measure of the likelihood that an event will occur. This precise definition helps mathematicians communicate clearly and avoid misunderstandings. As Landau (2001, p. 65) said such definitions “are imposed on the basis of expert advice” with the goal of “ease and accuracy of communication between those versed in the language of science.”

In the case of linear algebra, the use of both definitions can be seen as follows:

- Extracted Definitions: These are the common, everyday understandings of terms. For example, a "matrix" is commonly understood as a rectangular array of numbers, much like a table or spreadsheet. Similarly, a "vector" is commonly understood as a quantity that has both magnitude and direction.
- Stipulated Definitions: In contrast, stipulated definitions provide precise meanings that are essential for clear communication in mathematics. In linear algebra, a "matrix" is specifically defined as a rectangular array of numbers arranged in rows and columns, which is used to represent linear transformations and to solve systems of linear equations. Similarly, a "vector" is precisely defined as an element of a vector space. It can be represented as an ordered list of numbers, known as components, which can be added together and multiplied by scalars according to specific rules. For example, in the vector space $\mathbb{R}^n$, a vector is an ordered n-tuple of real numbers.

Mathematics has a distinct language, as everyday words often have a different and precise meaning in a mathematical context (Edwards & Ward, 2004; Hillman, 2013). Along with Robinson and Landau, we think of mathematical definitions as stipulated, whereas most, "everyday language" definitions are extracted (Edwards & Ward, 2008).

Students often have different understandings of a concept's formal definition compared to its exact mathematical definition. In this regard, Edwards and Ward (2004) showed that many undergraduate mathematics students fail to understand the difference between extracted (everyday language) definitions and stipulated (mathematical) definitions and as a result, they suggested that math classes at all levels should talk more about what makes mathematical definitions special because mathematical definitions play a crucial role in how professional mathematicians work and communicate. But when it comes to beginners learning math, it's helpful to have a framework that explains how they understand and use these definitions. This framework is termed as Concept Image (Tall, 1992; Vinner, 1991). It helps describe how people think about math concepts and their formal definitions. Edwards & Ward (2008) defined concept image as “the set of all the mental pictures associated in one's mind with the name of a particular concept as well as all the properties that characterize them. The concept image may be incomplete or mathematically incorrect, and can include naïve, non-mathematical associations with the concept name” (p.224). Tall and Vinner (1981) defined concept image as follows: “We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures” (p. 152).

For instance, when hearing the word “table”, a picture of a certain table can be evoked in your mind. Experiences of sitting at a table, eating at a table, etc., can be evoked as well. You can recall that many tables are made of wood, most of them have four legs (Vinner, 1991).

Similarly, for a student, the concept image of a "derivative" may consist only of the mechanical process of finding the derivative using rules like the power rule or the product rule. As a result, when asked about the derivative of a function at a specific point, the student might focus solely on performing the calculation, rather than understanding it as the slope of the tangent line to the graph of the function at that point.

As in linear algebra, for a student the concept image of A^{-1} may consist only of the computation of the matrix. As a result, when he is asked whether a given matrix B is the inverse of a matrix A, he may calculate A^{-1} instead of multiplying AB (Berman & Shvartsman, 2016).

The case of Linear Algebra

Stewart and Thomas (2007) mentioned that students often find their first university linear algebra experience very challenging. Various research tells us that one of the main obstacles is the lack of students' understanding of the importance of formal definitions. They try to solve conceptual and procedural problems before they understand the related concepts. This causes distorted concept images which results in misconceptions (Berman & Shvartsman, 2016). As summed up by Carlson (1993, p. 39), "My students first learn how to solve systems of linear equations, and how to calculate products of matrices. These are easy for them. But when we get to subspaces, spanning, and linear independence, my students become confused and disoriented."

In the words of Berman & Shvartsman (2016), "One of the reasons of the students' difficulties with definitions may be the way of teaching mathematics in high school, which for didactical reasons, is often not done rigorously, Undergraduate students try to transfer their previous learning experience to university studies. This experience is based on their beliefs that they should remember some rules and solve a lot of problems, preferably from previous exams. In their opinion learning definitions or proofs is a waste of time."

There are multiple research papers in linear algebra which conclude that the formal way of reasoning is particularly important in a linear algebra course as it requires more abstract thinking than other undergraduate mathematical courses. As mentioned in Dorier et al. (2000): "The main criticisms made by the students toward linear algebra concern the use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics. It is quite clear that many students have the feeling of landing on a new planet and are not able to find their way in this new world. On the other hand, teachers usually complain of their students' erratic use of basic tools of logic or set theory" (p. 28).

The role of definitions thus becomes crucial in learning linear algebra, especially given the subject's abstract nature. Without a solid understanding of formal definitions, students struggle to grasp more complex concepts like subspaces, spanning sets, and linear independence. Definitions serve as the foundation upon which all other understanding is built. They provide the precise language needed to describe and work with these abstract concepts.

When students understand and use these definitions correctly, they can avoid misconceptions and develop a more coherent understanding of the subject. This precision helps them see the connections between different concepts and how they fit into the larger framework of linear algebra. For example, understanding the formal definition of linear independence is essential for grasping the concept of a basis in a vector space. Linear independence means that no vector in a set can be written as a linear combination of the others. This concept is critical when determining a basis, which is a minimal set of vectors that spans the entire vector space. Without a clear understanding of linear independence, students might struggle with identifying or constructing a basis, leading to confusion in more advanced topics like dimension, orthogonality, and diagonalization. Therefore, a firm grasp of definitions not only clarifies immediate problems but also lays the groundwork for future learning.

Understanding the formal definitions in linear algebra not only helps in grasping complex concepts but also bridges the gap between students' concept images and the precise mathematical terminology. Stipulated definitions provide the exact meaning of terms such as linear independence, basis, and vector space, helping students correct their often incomplete or incorrect, concept images. Extracted definitions might lead to misconceptions if not properly aligned with their mathematical counterparts. By emphasizing stipulated definitions, teachers can thus help students refine their concept images, ensuring a deeper and more accurate understanding of linear algebra. This approach not only clarifies immediate problems but also establishes a strong foundation for future mathematical learning.

Importance of definitions

The role of definitions in mathematics is critical to students learning outcomes as understanding of concepts is mainly acquired through their definitions in mathematics (Vinner, 1991). Hence, definitions play an inevitable role not just in the teaching and learning of linear algebra, but the complete mathematics.

Students must use stipulated definitions in order to become discourse insiders into the subject of mathematics, as using and understanding stipulated definitions is a characteristic of mathematicians (Buehl, 2017; Edwards & Ward, 2004). Stipulated definitions help students share ideas, explain their thinking, build strong arguments, and make logical conclusions. Furthermore, using stipulated definitions correctly is one way for students to demonstrate mathematical literacy (Hillman, 2013).

In contrast, mathematical terminology can often confuse students, and a more conceptual way of understanding content is needed (Vinner, 1991). Students need to connect mathematics to the real world in order to develop an understanding of a concept, as students have an easier time making meaning of words through personal connections, such as concept images (Bruun, Diaz, & Dykes, 2015). Therefore, extracted definitions are beneficial for students in forming a concept image, while stipulated definitions are necessary for understanding the content. Thus, teaching and learning of mathematics requires simultaneous need for both types of definitions in the classroom.

Definitions might not be very important for the understanding of everyday language. So, some students and teachers might not see the importance of learning definitions because they think problem-solving doesn't really need definitions. This attitude makes the teaching and learning process in mathematics considerably difficult due to distorted concept images. In everyday language, many words like “dog,” “car” or “happy” lack formal definitions, although dictionaries attempt to provide some explanation. Consider a sentence like "my happy dog enjoys chasing cars." One does not need to look up definitions because these common words are understood without them. However, in technical contexts, such as in mathematics, terms are precisely defined to avoid confusion. So, if one is dealing with technical subjects, consulting definitions is crucial to prevent errors. For example, as mentioned by Vinner (1991), understanding a sentence like "among all rectangles with the same perimeter, the square has the largest area" requires consulting definitions to understand its meaning accurately. Note that in everyday life contexts, a square is not considered as a rectangle by most of the people, whereas in all mathematical contexts a square is a rectangle.

Conclusion and Recommendations

Berman & Shvartsman (2016) conducted an experiment among the students of linear algebra course. The research goal was based on two questions:

i.) Can the students be educated to appreciate the importance of definitions?

ii.) Does the understanding of definitions contribute to the students’ understanding of the studied material?

As mentioned in the paper, the experiment involved slightly changing the homework questions and the way of assessment. The procedural questions given as homework were replaced by theoretical questions on formal concept definitions and their basic properties. The experiment's results showed that these minor adjustments significantly impacted the students' learning attitudes. They enhanced the students' approach to the learning process. The students showed a keen interest in the details of definitions and theorems. It was evident that they became more engaged during lectures, asking more frequent and serious questions. Some students reported a change in their problem-solving approach: they first focused on understanding the definitions and properties of the related concepts before attempting to solve the problems. Thus, it was concluded that students can be taught to recognize the importance of definitions. In the words of Berman & Shvartsman (2016) themselves, "The results of our experiment show that students' understanding of formal definitions contributed to their comprehension of the studied material and that their attitude toward studying linear algebra became more serious."

Building on the findings of Berman & Shvartsman (2016), several strategies can enhance the teaching of mathematics and shift students' attitudes positively. One key approach is having a greater emphasis on definitions and theoretical concepts within the curriculum. This can be achieved by including more theoretical questions into homework assignments, assessments and text books ensuring that students engage deeply with the foundational principles of the subject. By doing so, students can start to appreciate the significance of these concepts early on, seeing their relevance to more advanced topics and practical applications.

Furthermore, lectures should be structured to highlight the importance of formal definitions, demonstrating their practical applications and connections to problem-solving. Teachers can provide concrete examples showing how definitions help in understanding and memorizing theorems and proofs, helping students understand the value of these core concepts. Additionally, fostering an interactive classroom environment is crucial. Encouraging open discussions, promoting questions, and creating a collaborative learning atmosphere allows students to explore their understanding of definitions and theorems more effectively.

Providing continuous feedback that focuses not only on the correctness of solutions but also on understanding the underlying concepts is another important strategy. This approach helps students to appreciate the importance of mastering definitions and theoretical knowledge. Integrating technology, such as educational software that emphasizes conceptual understanding can also offer students alternative ways to engage with the material. By adopting these methods, teachers can help students develop a deeper appreciation for the rigor and beauty of learning mathematics. In conclusion, the importance of definitions in teaching and learning of linear algebra and mathematics cannot be overstated, as they form the foundation for a clear and precise understanding of mathematical concepts.

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