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Deterministic and Stochastic Holling-Tanner Prey-Predator Models

Bharat Bahadur Thapa[1, *], Samir Shrestha[2], Dil Bahadur Gurung[2]

[1]*Department of Science and Humanities, Cosmos College of Management and Technology, Lalitpur, Nepal*

[2]*Department of Mathematics, Kathmandu University, Kavre, Nepal*

Abstract

A modified version of the so called Holling-Tanner prey-predator models with prey dependent functional response is introduced. We improved some new results on Holling-Tanner model from Lotka-Volterra model on real ecological systems and studied the stability of this model in the deterministic and stochastic environments. The study was focused on three types of stability, namely, stable node, spiral node, and center. The numerical schemes are employed to get the approximated solutions of the differential equations. We have used Euler scheme to solve the deterministic prey-predator model and we used Euler-Maruyama scheme to solve stochastic prey-predator model.

Keywords: *Holling-Tanner model; Prey; Predator; Stability analysis; Brownian motion.*

1. Introduction

Most of the research papers or research activities of mathematics are found in the field of mathematical biology and ecology. The area of bio-mathematical research is growing fast day by day and its application raises the concern about the preservation of the ecological balance in nature. There is first kind of animal predator that survives by eating another animal prey which in turn survives by eating some third item available in the nature. It is well known by everyone that there is a constant struggle for survival among different species animal living in the same environment. The most common examples of prey and predator are zebra and lion, fish and shark, deer and leopard.

The first mathematical model of population study was developed by Robert Malthus (1766-1834) in 1798 which is an exponential model that considers only one species under natural abundance of the food and other things for existence. A Belgian Mathematician Pierre Francois Verhulst (1804 – 1849), in 1838 suggested

[*]Corresponding author
Email: bharatbahadurthapa@cosmoscollege.edu.np
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a logistic model for single population with the per capita growth rate, carrying capacity of available food resource. The objectives of the paper are to analyze the dynamical behavior of Holling-Tanner prey- predator model. Deterministic Holling- Tanner prey-predator model is well known model for studying the interaction of prey and predator species. Deterministic model could not capture the exact variability in nature Then, Stochastic models are required for an accurate approximation of the dynamics of such interactions. After non-dimensionalization of the model, we analyze stability of equilibrium position with the help of determinant, trace and discriminant of the model. We have used Euler scheme to solve deterministic model equations and Euler-Maruyama scheme is implemented to solve the stochastic model equations. The numerical solutions are presented graphically.

2. Holling-Tanner Model Equation

The Holling-Tanner model for prey-predator interaction is the modification of the Leslie-Gower model where the per capita capturing rate is replaced by a saturation function of prey population, which is known as Holling type II functional response. Functional response is defined as the temporal rate at which an individual predator kills prey, and it is the average number of preys killed per predator per unit time. The Holling-Tanner model is given by system of ordinary differential equations (ODEs) [4]

$$\frac{dX}{dT} = r_1 X\left(1 - \frac{X}{K}\right) - \frac{mXY}{B + X}, \qquad (1)$$

$$\frac{dY}{dT} = r_2 Y\left(1 - \frac{Y}{hX}\right).$$

Where,

$X(T)$ is the size of prey at time T,

$Y(T)$ is the size of predator at time T,

r_1, r_2 are intrinsic growth rates,

K is prey carrying capacity,

hX is predator carrying capacity, h is the number that control prey as food for predator to survive and h< 1.

m is maximum number of prey that can be eaten by per predator per unit time,

B is the half saturation constant (prey number),

$\frac{mX}{B+X}$ is the predation rate.

By setting the following substitutions

$$T = \frac{t}{r1},\ X = Kx,\ Y = hKy$$
$$\Rightarrow dT = \frac{dt}{r1},\ dX = Kdx,\ dY = hKdy\,.$$

Then, dimensional ODEs in (1) changed in to the non-dimensional form as follows

$$\frac{dx}{dt} = x(1 - x) - \alpha\frac{xy}{b + x},$$

$$\frac{dy}{dt} = \beta y\left(1 - \frac{y}{x}\right) \qquad (2)$$

where,

$$\alpha = \frac{mh}{r_1}, \qquad \beta = \frac{r_2}{r_1}, \qquad b = \frac{B}{K}.$$

The co-existence equilibrium point of non-dimensional differential equations (2) is

$$(x_*, y_*) = \left(\frac{1-(\alpha+b) + \sqrt{D}}{2}, \frac{1-(\alpha+b) + \sqrt{D}}{2}\right),$$

Where

$$D = (\alpha + b - 1)^2 + 4b \ \geq 0.$$

Equilibrium point depends on the value of a and b.

We discuss the stability analysis of the non-dimensional form (2) of Holling-Tanner equations by finding the Jacobian matrix of the system of equations and computing its determinant, trace and discriminant.

For that, letting from (2)

$$f = x - x^2 - \frac{\alpha xy}{b+x} \text{ and } g = \beta y - \frac{\beta y^2}{x}$$

Taking partial derivative of and with respect to and, we get the Jacobian matrix of the system at the equilibrium point) (x_*, y_*) is

$$A = \begin{bmatrix} f_x(x_*, y_*) & f_y(x_*, y_*) \\ g_x(x_*, y_*) & g_y(x_*, y_*) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 - 2x_* - \dfrac{\alpha b x_*}{(b + x_*)^2} & -\dfrac{\alpha x_*}{b + x_*} \\ \beta & -\beta \end{bmatrix}$$

The trace *tr (A)* of Jacobian matrix is

$$\Delta_1 = -\beta + 1 - 2x_* - \frac{\alpha b x_*}{(b + x_*)^2}.$$

and the determinant *det(A)* of Jacobian matrix A is

$$\Delta_2 = \beta\left(2x_* + \frac{\alpha b x_*}{(b+x_*)^2} + \frac{\alpha x_*}{b+x_*} - 1\right).$$

Similarly, the characteristic equation of the Jacobian matrix A is

$$\delta^2 - \delta\Delta_1 + \Delta_2 = 0,$$

where δ denotes the eigen-value of the matrix A . The discriminant Δ of the characteristic equation is given by

$$\Delta = \left(-\beta + 1 - 2x_* - \frac{\alpha b x_*}{(b + x_*)^2}\right)^2 - 4\beta\left(2x_* + \frac{\alpha b x_*}{(b + x_*)^2} + \frac{\alpha x_*}{b + x_*} - 1\right)$$

Theorem1: Assume that (x_*, y_*) is an equilibrium point, then the following relation hold [2]:

I. If $\Delta_2 < 0$, then (x_*, y_*) is a saddle point.

II. If $\Delta_2 > 0$ and $\Delta_1^2 - 4\Delta_2 \geq 0$, then (x_*, y_*) is a node. It is a stable if $\Delta_1 < 0$ and unstable if $_1 > 0.\Delta$

III. If $\Delta_2 > 0$ and $\Delta_1^2 - 4\Delta_2 < 0$, then (x_*, y_*) is a focus. It is a stable if $\Delta_1 < 0$ and unstable if $_1 > 0\Delta$

IV. If $\Delta_2 > 0$ and $\Delta_1 = 0$, then (x_*, y_*) is a center of focus.

2.1 Graphical Analysis of the Determinant, Trace and Discriminant of the Jacobian Matrix

Reference to the Theorem 1, we need positive determinant and negative trace for stability of equilibrium point.

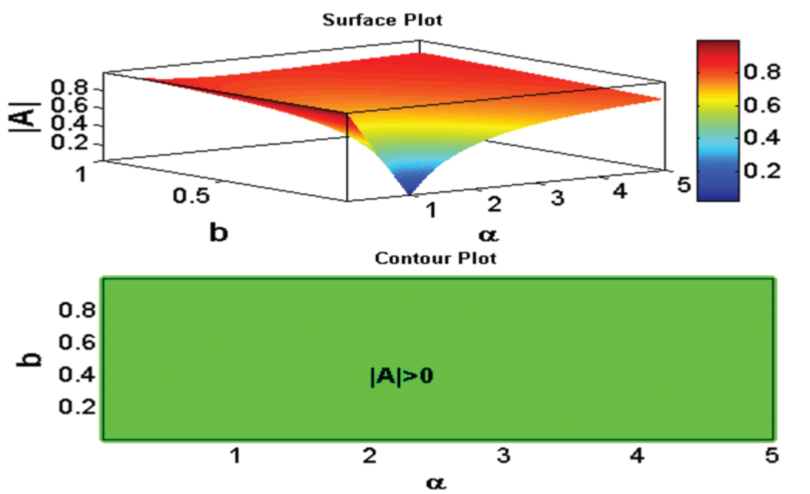

Figure 1: The determinant of the Jacobian matrix A that shows the positivity

Figure 1 shows surface plot and contour plot of the determinant of the Jacobian matrix for various values of and. The determinant of the Jacobian matrix that shows the positivity which ensured by the contour plot (green color) in the Figure 1

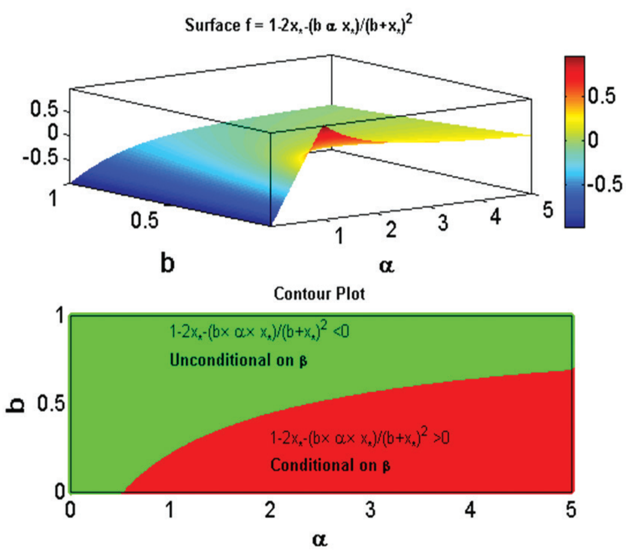

Figure 2: Trace of the Jacobian matrix whose negativity is unconditional and conditional on β

Figure 2 shows the surface plot and contour plot of the trace of Jacobian matrix for different values of α and b. The green region shows the negative trace which is unconditional on β and the red region represents the positive trace which is conditional on β.

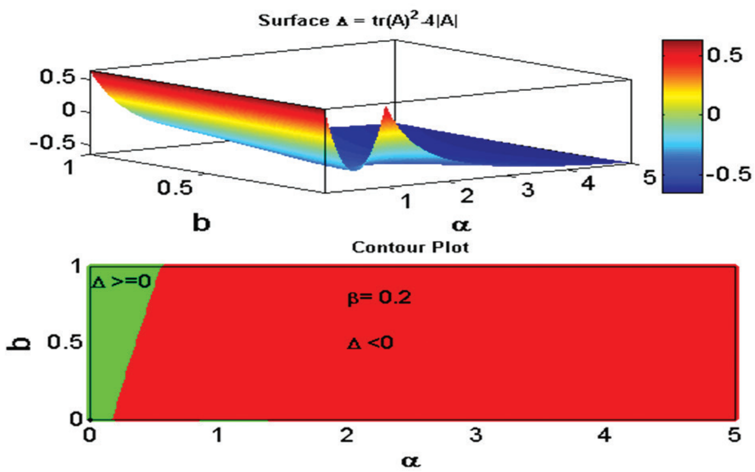

Figure 3: Surface and contour plot of the discriminant Δ *for* $\beta = 0.2$. $\Delta \geq 0$ *in the green contour and* $\Delta \leq 0$ *in the red contour region*

Figure 3 shows the surface plot and contour plot of the discriminant of the Jacobian matrix for various values of and When the surface and contour (green and red color) plot of the discriminant is presented in the Figure 3. In the contour plot, the green region shows the discriminant is positive and the red region shows that the discriminant is negative.

The different types of stable equilibrium point that depends on the nature of discriminant of the characteristic equation of Jacobian matrix. If the discriminant is positive, eigen values of the Jacobian matrix real and negative that implies the equilibrium point is asymptotically stable. If the discriminant is negative, the eigen values are complex root with real part negative that implies the equilibrium point to be spirally stable. If the discriminant is zero, circular limit cycle exists.

3. Stochastic Holling-Tanner Model Equation

In fact, biological systems are random in nature i. e. our environments are stochastic in nature. The noise plays important role in the structure and function of such system [7]. An important concept in stochastic modeling is that of a Wiener process. A Wiener process is a continuous time stochastic process which is named of Nobert Wiener. It is also known as the standard Brownian motion after Robert Brown.

To formulate the stochastic model, we will perturb intrinsic growth rate of both population by white noise term ξ_1 and ξ_2. The stochastic Holling-Tanner model is given by

$$\frac{dX}{dT} = r_1 x\left(1-\frac{X}{K}\right) - \frac{m\,X\,Y}{B+X} + \mu_1 X \xi_1(T)$$

Using non-dimensionalization setting as done before, we have

$$dx = \left[x(1-x) - \frac{\alpha\,x\,y}{b+x}\right]dx + \sigma_1 x dW_1 \tag{3}$$

where, $\alpha = \frac{mh}{r_1}$, $\sigma_1 = \frac{\mu_1}{r_1}$

similarly, $\frac{dY}{dT} = r_2 Y \left(1 - \frac{Y}{hX}\right) + \mu_2 Y \xi_2(T)$

Using non-dimensionalization setting, we have

$$dy = \beta\, y \left(1 - \frac{y}{x}\right) + \sigma_2\, y\, dW_2 \tag{4}$$

where

$$\beta = \frac{r_2}{r_1}, \sigma_2 = \frac{\mu_2}{r_1}, b = \frac{B}{K}$$

and W_1 and W_2 are the Brownian motion (Wiener process) [8].

4. Numerical Result of Holling -Tanner Prey-Predator Model

We have used Euler's method to solve the non-dimensional deterministic prey-predator model equation (2). To solve the stochastic differential equations (3) and (4), the Euler-Maruyama scheme is implemented. The study focused on three types of stability, namely, stable node, spiral node and center. We have presented numerical study on the dynamic of the population based these three different types of stabilities of the equilibrium points by choosing the appropriate arbitrary parameter values.

Result 1:

Choosing the parameters values $r_1 = 0.6$, $r_2 = 0.2, k = 100$, $m = 20, B = 50$, $h = 0.2, \alpha = 0.3408$, $b = 0.3969, \beta = 0.2, x_0 = 0.6, y_0 = 0.4, \sigma_{1,} = \sigma_2 = 0.01$

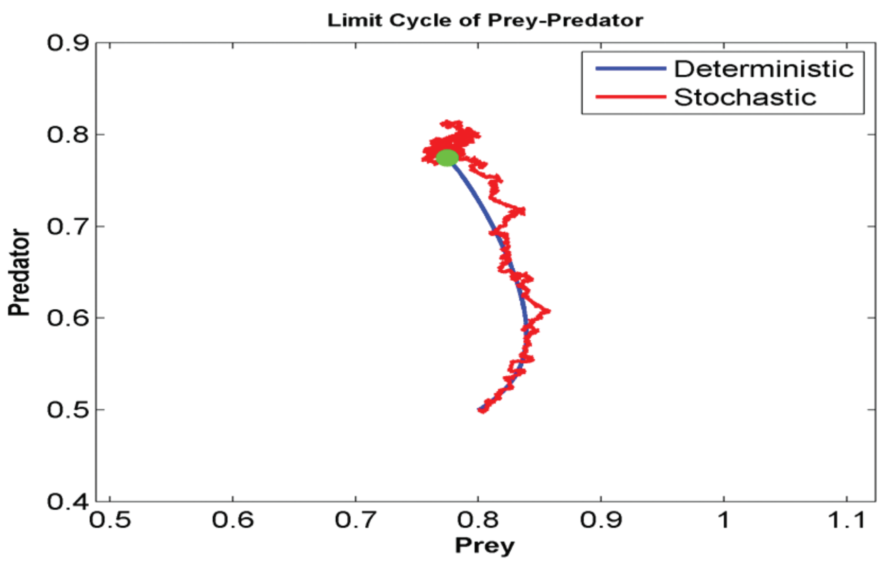

Figure 4: The limit cycle of the stable node.

Figure 4 represents the limit cycle of the prey and predator population. From parameters mentioned above, we get positive determinant, negative trace and hence eigenvalues are real and negative. Equilibrium point $(x_*, y_*) = (0.7747, 0.7747)$. According to Theorem 1, it proves that it is stable node.

Result 2:

Choosing the parameters values $r_1 = 0.6$, $r_2 = 0.2$, $k = 100$, $m = 20$, $B = 50$, $h = 0.2$, $\alpha = 0.8$, $b = 0.1$, $\beta = 0.2$, $x_0 = 0.8$, $y_0 = 0.5$, $\sigma_1 = \sigma_2 = 0.01$..

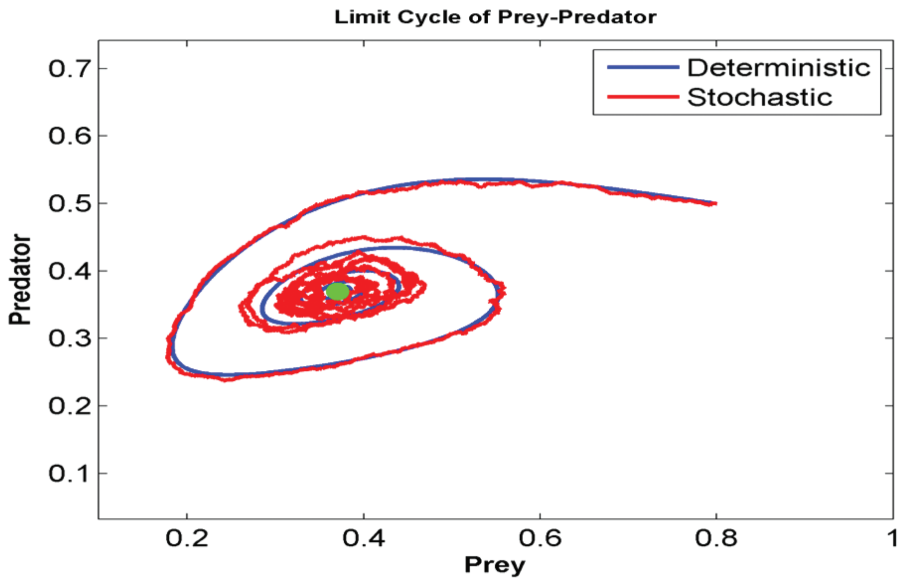

Figure 5: The limit cycle of the spiral stability of the equilibrium point.

Figure 5 shows that the limit cycle of prey and predator population. We get positive determinant, negative trace and hence complex eigenvalues with real part is negative. Equilibrium point $(x_* \ y_*) = (0.3702, 0.3702)$. By Theorem 1, it proves that it is spiral stability.

Result 3:

Choosing the parameters values $r_1 = 0.6, r_2 = 0.2, k = 100$, $m = 20$, $B = 50$, $h = 0.2$, $\alpha = 0.8952, b = 0.1$, $\beta = 0.2$, $x_0 = 0.3$, $y_0 = 0.3, \sigma_1 = \sigma_2 = 0.01$.

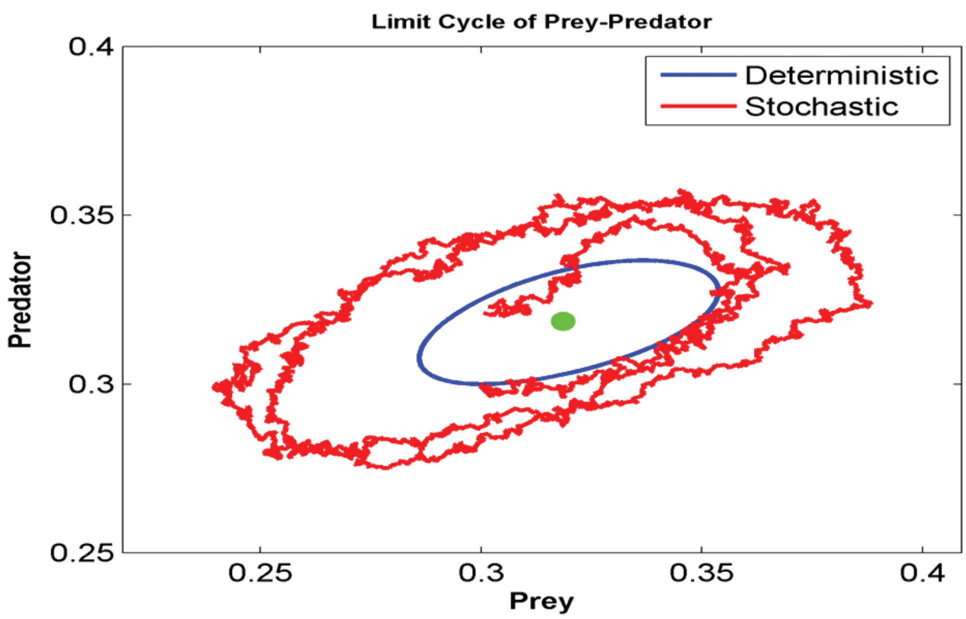

Figure 6: The limit cycle of the circular stability of the equilibrium point.

Figure 6 denotes the limit cycle of prey and predator of the circular stability. From above data, we get positive determinant, zero trace and hence all eigenvalues are purely imaginary. Equilibrium point $(x_*, y_*) = (0.3186, 0.3186)$. By Theorem 1, it proves that it is circular stability.

5. Conclusion

From our paper, we have determined the global dynamics of prey dependent modified Holling-Tanner prey-predator model. The dynamics of the interaction between the prey and their predator is independent of the growth rate of the predator population and the abundance of additional food available for the predator and both the spices are capable to maintain their equilibrium levels of the initial population densities. We analyzed Holling-Tanner prey-predator model in deterministic and stochastic parts. A stochastic model provides a more realistic picture of a natural system than its deterministic model. Finally, we have numerically studied the stability of Holling-Tanner prey-predator model in terms of the stable, spiral, and center nodes by choosing suitable parameters values.

Conflict of Interest

Not declared by the authors.

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