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EXPLORING C^*-ALGEBRA: CENTRAL SEQUENCES AND SOME RESULTS

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ABSTRACT

We study central sequences of C^*-algebras and find associations of the central sequences of a C^*-algebra and its depictions with applications. Overall, results on central sequences are crucial for understanding the algebraic and topological properties of various mathematical structures, especially in functional analysis and operator theory.

In order to understand the structure and characteristics of C^*-algebras, central sequences are essential. In this paper, we investigate the notion of core sequences in C^*-algebras and talk about several important conclusions associated with them. First, we give a definition of central sequences and show how crucial they are to the theory of C^*-algebras. We then discuss Kadison's Central Sequence Theorem, a basic result that uses central sequences to characterize several characteristics of C^*-algebras. We also go over the uses of central sequences in functional analysis, operator theory, and quantum physics, among other branches of mathematics. This article attempts to give a greater knowledge of central sequences and their function in the theory of C^*-algebras through a examination and accompanying results.

A key concept in C^*-algebra theory are central sequences, which clarify the complex interactions between geometric features, algebraic structure, and functional analysis. Mathematicians and physicists are gaining new insights into the properties of C^*-algebras through the study of central sequences and the results that go along with them. This is contributing to our understanding of these fundamental mathematical objects and the wide range of applications they have in other fields.

Key words: central sequence, *C**-algebra, unital, trivial central sequence, hyper-central sequence **MSC 2020**: 14C35,14J70,22D25,46L05

INTRODUCTION

Central sequences play a crucial role in the study of *C**-algebras, providing valuable insights into their structure and properties. In this article, we explore the concept of central sequences in *C**-algebras and discuss some significant results related to them. We begin by defining central sequences and illustrating their importance in the theory of *C**-algebras. Subsequently, we delve into Kadison's Central Sequence Theorem, a fundamental result that characterizes certain properties of *C**-algebras using central sequences. Furthermore, we discuss applications of central sequences in various areas of mathematics, including operator theory, functional analysis, and quantum mechanics. Through a comprehensive analysis of central sequences and associated results, this article aims to provide a deeper understanding of their role in the theory of *C**-algebras.

*C**-algebras, a cornerstone of operator algebra theory, represent a powerful mathematical framework with diverse applications across various fields including functional analysis, quantum mechanics, and mathematical physics. Central sequences are vital to the study of *C**-algebras and are also comprehending the structure and characteristics of these algebraic systems (Omland, 2013).

*C**-algebras are complex algebraic structures with involution operations and norms, connected to mathematical structures like groups and topological spaces. They admit concrete representations as bounded operators on Hilbert spaces. The Gelfand-Naimark theorem is a fundamental result in *C**-algebra theory. *C**-algebras are essential in non-commutative geometry, theoretical physics, and unifying quantum mechanics and general relativity, playing a central role in modern mathematical and physics (Omland, 2013).

Understanding *C**-Algebra

*C**-algebras are a fundamental concept in functional analysis and operator theory, with applications in areas such as quantum mechanics, quantum field theory, and signal processing. Let's break down some key points to understand them better:

Definition: A *C**-algebra is a complex algebraic structure equipped with an involution operation and a norm that makes it a Banach algebra.

Formally, a C^*-algebra A over the complex numbers is a complex algebra with an involution $*$ and a norm $\|\cdot\|$ such that:

$\|ab\| \leq \|a\|\|b\|$ (the norm is sub multiplicative),

$\|a^* a\| = \|a\|^2$ for all a in A.

The involution operation $*$ on a C^*-algebra is an operation that associates with each element a in the algebra an element $a*$ such that:

$(ab)^* = b^* a^*$ for all a, b in A.

$(a^*)^* = a$

$(a+b)^* = a^* + b^*$

$(\alpha a^*)^* = \alpha - a^*$ for all scalars α.

Examples

The space B(H) of bounded linear operators on a Hilbert space H is a C^*-algebra with the usual operations of addition, multiplication, scalar multiplication, and the adjoint operation. The algebra C(X) of continuous complex-valued functions on a compact Hausdorff space X is a commutative C^*-algebra with pointwise operations and the complex conjugate as the involution. Understanding C^*-algebras involves exploring their structure, properties, and various examples, along with their applications in different areas of mathematics and physics.

C^*-algebra is a multifaceted algebraic structure furnished with a norm that satisfies certain properties, including completeness and the involution operation. These algebras aid as a natural generalization of Hilbert space. The algebra of continuous functions, and algebra of a Hilbert space are examples of it (Elliot, 1973).

Central Sequences

Central sequences are a crucial notion within the theory of von Neumann algebras, which are a special type of C^*-algebra. Given a sequence (x_n) in a von Neumann algebra M, the sequence is said to be a central sequence if it satisfies the following property:

$\|[x_n, y]\| \to 0 \to 0$ for all y in M.

Here, $[\cdot,\cdot]$ denotes the commutator of two operators, and $\|\cdot\|$ represents the operator norm.

Central sequences provide a way to understand the behavior of sequences of operators within von Neumann algebras. They often arise in the study of properties such as amenability, nuclearism, and type classification of von Neumann algebras. Central sequences are extensively used in the theory of von Neumann algebras, particularly in the study of their structure and classification. They are crucial in characterizing various classes of von Neumann algebras and understanding their operator-theoretic properties (Omland, 2013).

Central sequences are sequences of operators within von Neumann algebras that commute (in a strong sense) with all elements of the algebra, and they are essential for studying the structure and properties of these algebras.

Central sequences are sequences in a C^*-algebra that converge weakly to zero and commute with every element in the algebra. Informal terms, C^*-algebra of sequence $\{x_n\}$ is central if it converges to zero in norm for each element in A. Central sequences provide valuable insights and mechanisms of C^*-algebras, shedding light on their symmetries, spectral properties, and geometric features (Omland, 2013).

PROBLEM OF STATEMENT

Categorizing C^*-algebras and figuring out C^*-algebraic invariants have long piqued the curiosity of operator algebraists. Some relationships between automorphism groups of C^*-algebras, their derivations and central sequences have been established in the past. Central sequences in C^*-algebras are sequences of elements that converge to zero in a particular sense (Weaver, 2003). Understanding their behavior and properties is crucial in many contexts, including the study of amenability, unclarity, and simplicity of C^*-algebras. Analyzing results related to central sequences involve in proving theorems or establishing connections between central sequences and other aspects of C^*-algebras theory, such as K-theory, crossed products, or von Neumann algebras (Omland, 2013).

LITERATURE REVIEW

Sakai (1971) proved that an inner product is the result of every derivation of a simple unital C^*-algebra. According to Elliot (1973), all C^*-algebras have interior derivations. Research by Akemann and Pedersen (1979) shows that a separable C^*-algebra is a continuous if and only if each central sequence is trivial. They went on to define it as the direct sum of a continuous trace C^*-algebra and simple C^*-algebras.

Phillips (1988) explained that an uncountable outer automorphism group exists for a specific C^* algebra. He also investigated the connection between central sequences and the automorphism group.

Dixmier (1977) gave the first description of continuous trace C^*-algebras. It is difficult to verify his definition and even more difficult to recall. It is also not easy to ascertain if, for a given C^*-algebra, every central sequence is trivial.

Several mathematicians like Ando & Kirchberg (2016), Blackadar (2006), Brown & Ozawa (2008), Olsen & Pedersen (1989), Enders, D. & Shulman (2022) etc. have contributed and enriched the theory of C^*-algebra with their applications in different spaces.

Literatures concerning this theory can be found in any standard text books, reference book and monographs of C^*- algebra, for instance we refer a few; Strung & Parera (2021), Davidson (1996), Willard (1968), Kadison & Ringrose (1986), Padersen (1979) etc.

PRELIMINARIES

*C**-algebra

Banach algebra A is a function denoted by $x \mapsto x^*$ satisfied the following axioms:

$(x + y)^* = x^* + y^*$

$(xy)^* = y^* x^*$

$(x^*)^* = X$

$\|x^*\| = \|x\|$ for all x, y $\in A$ and $\lambda \in \mathbb{C}$.

$$\|x^* x\| = \|x x^*\| = \|x\|^2 \tag{1}$$

Central Sequence

A C^*-algebra is called a Central Sequence for $x \in A$ given by

$\|a_n x - x a_n\| \to 0$ for $n \to n$

Example

$(a_n) \subset A$ is a sequence converging to 0.

Hyper central sequence

A central sequence is hyper-central for any

$$\|a_n b_n - b_n a_n\| \to 0, \text{ as } n \longrightarrow 0 \tag{2}$$

METHOD AND DISCUSSION

This work will analyze core sequences in some unital, separable C^*-algebras and connects them to"completely continuous representation" C^*-algebras, which are multiplicity free representations in unital C^*-algebras. The work also establishes the multiplicity-free and point norm closure of a representation in the spectrum of a C^*-algebra.

Results on Central Sequences

When discussing results on central sequences, it typically involves studying the behavior of sequences in relation to certain structures like Banach algebras, C^*-algebras.

Theorem 1. All central sequences for a C^*-algebra A must also be hyper-central if they are trivial.

Proof. Assume all central sequence is trivial. Let (a_n) is a sequence and is hyper-central and (b_n) be any central sequence. Now, (a_n) is trivial

$$(a_n) = (z_n + w_n)$$

$$b_n(z_n + w_n) - (z_n + w_n)\, b_n\| \leq \| b_n z_n - z_n b_n \| + \| b_n w_n - w_n b_n \quad (3)$$

The second sum on the R.H.S. tends to zero and we get

$$\| b_n z_n - z_n b_n \| \leq \| b_n z_n \| + \| z_n b_n \| \quad (4)$$

Now (b_n) is bounded where (z_n) is null convergent. This shows that (a_n) is hyper-central.

Theorem 2. Prove that every central sequence of equation (4) is trivial.

Proof. Let $(f_n) = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ is a central sequence in A. Then (f_n) should be asymptotically commutative,

$$\text{P(t)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \text{q(t)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\lim_{n\to 0} \| f_n p - p f_n \| = 0$$

$$\lim_{n\to 0} \left\| \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right\| = \lim_{n\to 0} \left\| \begin{matrix} o & -b_n \\ c_n & 0 \end{matrix} \right\| = 0 = \lim_{n\to 0} \| f_n q - q f_n \|$$

$$\lim_{n\to 0} \left\| \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right\| = \lim_{n\to 0} \left\| \begin{matrix} -c_n & a_n - d_n \\ 0 & c_n \end{matrix} \right\|$$

Where $b_n \to 0,\ c_n \to 0,\ d_n - a_n \to 0\ as\ n \to \infty$

$$f_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} 0 & b_n \\ c_n & d_n - a_n \end{pmatrix} + \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix} \tag{5}$$

KADISON’S CENTRAL SEQUENCES THEOREM

Kadison's Central Sequence Theorem is a significant result in the theory of von Neumann algebras, named after Richard V. Kadison. This theorem provides a characterization of certain properties of von Neumann algebras using central sequences. The theorem provides a powerful tool for characterizing properties of von Neumann algebras in terms of the behavior of sequences of operators. It highlights the significance of central sequences in understanding the structure and properties of von Neumann algebras, particularly in the context of factors and semi-finite von Neumann algebras. The equivalence between central sequences and approximately Hermitian sequences gives a concrete criterion for verifying central sequences, which can simplify the analysis of von Neumann algebras.

Kadison's Central Sequence Theorem establishes a deep connection between central sequences and certain properties of von Neumann algebras, offering insights into their structure and behavior. One of the influential results concerning central sequences is Kadison's Central Sequence Theorem, which declares that every separable C^*-algebra contains a non-trivial central sequence. This result underscores the ubiquity and significance of central sequences in the theory of it, highlighting their presence in highly non-commutative settings (Pedersen, 1979).

APPLICATIONS TO TOPOLOGICAL DYNAMICS

C^*-algebras are an essential area of study within functional analysis and operator theory, with applications in various branches of mathematics and theoretical physics. Central sequences play a crucial role in the theory of C^*-algebras, particularly in understanding their structure and properties. Central sequences have profound implications in the realm of topological dynamics. By analyzing central sequences associated with these groups, researchers have gained deeper insights into their structure, rigidity, and chaotic behavior, paving the way for advancements in ergodic theory and topological entropy (Weaver, 2003).

Quantum Statistical Mechanics

In quantum statistical mechanics, central sequences emerge as key objects of interest, offering a means to characterize equilibrium states and phase transitions in quantum systems. The analysis of central sequences provides a powerful tool for probing the thermodynamic properties and quantum fluctuations of physical systems described by C^*-algebras (Murphy, 2003).

Operator Algebras

C^*-algebras are a special class of Banach algebras with involution, and they generalize the concept of operator algebras. Many operator algebras arising from the study of differential equations, group representations, and quantum mechanics can be viewed as C^*-algebras. Central sequences are often used to distinguish between different classes of operator algebras and to study their structural properties.

Non-commutative Geometry

Central sequences are intimately related to the concept of non-commutative geometry, which studies spaces where the commutative property of multiplication does not hold. C^*-algebras serve as the algebraic framework for non-commutative geometry, providing tools to analyze geometric and topological properties of non-commutative spaces. Central sequences play a significant role in understanding the geometric structure of C^*-algebras and their associated non-commutative spaces.

K-theory and Index Theory

K-theory is a branch of mathematics that studies vector bundles and their associated algebraic structures. Central sequences are utilized in K-theory to define certain invariant classes associated with C^*-algebras, such as K-homology classes. These classes play a crucial role in index theory, which studies the relationship between operators on Hilbert spaces and topological invariants of spaces.

Classification of C-algebras*

Understanding the structure and classification of C^*-algebras is a central theme in the theory of operator algebras. Central sequences often provide important information about the structure of C^*-algebras and can be used to distinguish between different classes of C^*-algebras. Classifying

C^*-algebras based on their central sequences has led to significant advancements in the field.

Automorphism Groups

Central sequences also have applications in the study of automorphism groups of C^*-algebras. They help in characterizing the structure of automorphism groups and understanding their actions on the underlying C^*-algebra. Central sequences provide insights into the rigidity and flexibility of automorphism groups of C^*-algebras.

Overall, the study of central sequences and their applications in the theory of C^*-algebras is a rich and active area of research with connections to various branches of mathematics and theoretical physics.

ADVANCEMENT OF FIELD OF RESEARCH

Researchers are exploring deeper connections between C^*-algebras and geometric objects, pushing the boundaries of our understanding of non-commutative spaces and their geometric properties. Efforts in classifying C^*-algebras have intensified, with researchers developing new techniques to distinguish between different classes of C^*-algebras. Central sequences remain a valuable tool in this endeavor, but new invariants and methods are also being developed to tackle the classification problem from various angles. The study of topological phases of matter has emerged as a prominent application of operator algebras, particularly in condensed matter physics. Researchers are investigating how C^*-algebraic techniques can be used to understand and classify different topological phases, providing insights into the behavior of quantum systems.

The study of quantum groups and quantum symmetries involves algebraic structures that generalize traditional symmetries in quantum mechanics. C^*-algebras provide a natural framework for studying these structures, and recent advancements have led to a deeper understanding of the interplay between quantum groups, operator algebras, and non-commutative geometry. Operator algebras, including C^*-algebras, continue to interact with a wide range of mathematical disciplines, including representation theory, harmonic analysis, and mathematical physics. These interactions foster cross-pollination of ideas and techniques, driving further progress in the field.

CONCLUSION

In order to understand C^*-algebras and their structure and features, central sequences are essential. The idea of core sequences in C^*-algebras is examined in this article, along with some important findings about them. Central sequences are defined first, and their significance in the theory of C^*-algebras is demonstrated. After that, we explore Kadison's Central Sequence Theorem, a basic conclusion that uses central sequences to characterize several characteristics of C^*-algebras. In addition, we go over the uses of central sequences in operator theory, functional analysis, and quantum physics, among other mathematical disciplines. This paper had explained central sequences and related findings in greater detail, with a focus on their function in the theory of C^*-algebras.

Central sequences represent a fundamental aspect of C^*-algebra theory, elucidating the intricate interplay between algebraic structure, functional analysis, and geometric properties. Through the exploration of central sequences and their associated results, mathematicians and physicists continue to uncover new insights into the nature of C^*-algebras, enriching our understanding of these foundational mathematical objects and their myriad applications across diverse domains.

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