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CONVERGENCE OF INFINITE PRODUCT BY SCHLÖMILCH'S METHOD

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ABSTRACT

This paper presents the resolution of *sine* and *cosine* as an infinite product from Schlomilch method. It also discusses the convergence of the infinite products of sin θ and cos θ. Some important Series from Infinite Product has been evaluated.

Keywords: infinite product - schlömilch's method - convergence - divergence - semi-convergent

INTRODUCTION

This method was derived by Oscar Xavier Schlomilch in 1857. This method is used for a sum of series in terms of Bessel function which is based on the summation of trigonometric series (Krantz 1999).

Schlömilch's introduced the series of the form

$$\mathfrak{S}_v\left(z\right) := \sum_{n=1}^{\infty} \mathcal{L}_n J_V\left(\left(v+n\right)z\right), \text{z} \in \mathbb{C}$$

where v, $\mathcal{L}_n$ are constants and Jv is the Bessel function of the first kind of order v. This kind series are called Schlömilch's series (Schlomilch, 1846). Rayleigh has proved that such series has active roles in physics, because for $v = 0$, they are useful in investigation of a periodic transverse vibrations uniformly distributed in direction through the two dimensions of the membrane (Rayleigh 1911). Schlomilch series present various features of purely mathematical interest and it is remarkable that a null–function can be represented by such series in which the coefficients are not all zero (Watson 1966).

Trickovic, Stankovic and Vidanovic described that functions are suitable for applications of the theory of functions of complex variable. They gave a huge scope for the application than provided by trigonometric functions in the theory of fourier series. The theory of these functions have been used for solving some problems in acoustics, hydrodynamics, radio physics, nuclear physics (Trickovic, Stankovic &Vidanovic 2019).

Stankovic, Vidanovic & Trickovic have taken trigonometric series in terms of the Riemann zeta function and related functions of reciprocal powers. These reciprocal power and Riemann zeta functions are evaluated by means of closed form formulae. These closed formula help them to find the sums of some Schlömilch's series (Stankovic, Vidanovic & Trickovic 2003).

Miller used poison summation formula to evaluate cosine transformof the hyper geometric function, and also evaluated several new Sclömilch's series (Millar 1997). Jankov and Pogany described integral representations for Schlömilch's series of borel functions of first kind $J\nu$ with the help of first kind Kapteyn-type series (Jankov & Pogany 2012).

Several mathematicians Twersky, Miller, Bondarenko etc. have made their contribution and enriched the theory of infinite product and Schlömilch's method and their applications in different directions. Literature concerning infinite product and summation of some trigonometric series and Schlömilch's method can be found in any standard textbooks and monographs, for instance, (Agnew & Walker 1947, Arfken 1985, Hyslop 2006).

Statement of the Problem

Infinite products are the background for the development of expansion of function. Many works have been done on convergence of infinite product by Schlömilch's series but there are very work on convergence by Schlömilch's method for trigonometric function. So this research paper will fulfill this gap.

The major research problems are identified as convergence of infinite product by Schlömilch's method. To address the problem statement and the knowledge gaps described above, the following research questions are seeking to be answered.

- How sine and cosine function are resoluted as an infinite product?
- How infinite product converges by Schlömilch's method?

OBJECTIVE

The general objective of this research is to study the convergence of infinite product by Schlömilch's method. The specific objectives are as follows:

- To study there solution of sine and cosine function as an infinite product.
- To describe convergence of infinite product by Schlömilch's method.

Definition and Preliminary

Let u_n is any real function of n, defined for all positive integral numbers n then the product

$$F_n := u_1, u_2 \tag{1}$$

which is written with the use of symbol Π, signifying the product of the factors u_n as

$$F_n = \Pi_{r=1}^{n} u_n \tag{2}$$

Let $\lim_{n \to \infty} F_n$ is finite and non –zero then the infinite product converges to limit F and written as $\Pi_{n=1}^{\infty} u_n = F$ (3)

If F_n doesn't tend to finite real number then infinite product diverges for $u_n > 0$.

$$\text{Ln}(F_n) = \ln(\Pi_{r=1}^{n} u_r) = \Sigma_{r=1}^{n} \ln \quad u_r \tag{4}$$

which justifies that if F_n tends to zero, the series $\Sigma_{r=1}^{n} \ln \quad u_r$ tend to ∞. When F_n tends to zero, the infinite product diverges to zero (Asatryan 2015) .

Resolution of sine function and cosine function as an infinite product: Schlömilch method

The Trigonometric Ratios of angles 2A in terms of A

$$\sin 2A = 2\sin A.\cos A = 2\sin\frac{A}{2}.\cos\frac{A}{2} = 2\sin\frac{A}{2}.\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)$$

Then $$\sin A = 2\sin\frac{A}{2}.\sin\left(\frac{\pi}{2}+\frac{A}{2}\right) = 2\sin\frac{A}{2}.\sin\left(\frac{\pi+A}{2}\right)$$

$$\sin\theta = 2\sin\frac{\theta}{2}.\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}.\sin\left(\frac{\pi+\theta}{2}\right) \tag{5}$$

Substituting $\theta = \frac{\theta}{2}$ in equation (5)

$$\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2^2}\sin\frac{\pi+\frac{\theta}{2}}{2} = 2\sin\frac{\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}$$

and $\sin\frac{\pi+\theta}{2} = 2\sin\frac{\pi+\theta}{2^2}\sin\frac{\pi+\frac{\pi+2}{2}}{2} = 2\sin\frac{\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}$.

Substituting these value in equation (5), after rearranging them

$$\sin\theta = 2^3\sin\frac{\theta}{2^2}\sin\frac{\pi+\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}$$

Applying successively (5) for these four factors,

$$\sin\theta = 2^7\sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}\sin\frac{3\pi+\theta}{2^3}\sin\frac{4\pi+\theta}{2^3}\sin\frac{5\pi+\theta}{2^3}\sin\frac{6\pi+\theta}{2^3}\sin\frac{7\pi+\theta}{2^3}$$

$$= 2^{(2^3-1)}\sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\cdots\cdots\cdots\sin\frac{(2^3-1)\pi+\theta}{2^3}$$

= ... ...

$$= 2^{(2^n-1)}\sin\frac{\theta}{2^n}\sin\frac{\pi+\theta}{2^n}\cdots\cdots\cdots\sin\frac{(2^n-2)\pi+\theta}{2^n}\sin\frac{(2^n-2)\pi+\theta}{2^n}$$

$$= 2^{(2^p-1)}\sin\frac{\theta}{p}\sin\frac{\pi+\theta}{p}\sin\frac{2\pi+\theta}{p}$$

$$\cdots\cdots\cdots\cdots\sin\frac{(p-2)\pi+\theta}{p}\sin\frac{(p-1)\pi+\theta}{p} \qquad (6)$$

Here P= 2^n

Last factor in equation (6),

$$\sin\frac{(p-1)\pi+\theta}{p} = \sin\left(\pi - \frac{\pi-\theta}{p}\right) = \sin\left(\frac{\pi-\theta}{p}\right)$$

The last but one factor in equation (ii) is

$$\sin\frac{(p-2)\pi+\theta}{p} = \sin\left(\pi - \frac{2\pi-\theta}{p}\right) = \sin\frac{2\pi-\theta}{p},$$ and so on.

The $\left(\frac{p}{2}+1\right)$ th factor from the beginning in equation (6) is

$$\sin\frac{\left(\frac{p}{2}+1-1\right)\pi+\theta}{p}=\sin\frac{\frac{p\pi}{2}+\theta}{p}=\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)=\cos\frac{\theta}{p}$$

The number of factors in equation(6) is even because p is even. So, combining the second and the last factors, the third and the last but one and so on, but leaving along the first and the $\left(\frac{p}{2}+1\right)th$ (factors which without conjugates, the equation (6) transforms to (Twersky 1961).

$$\sin\theta=2^{p-1}\sin\frac{\theta}{p}\left\{\sin\frac{\pi+\theta}{p}.\sin\frac{\pi-\theta}{p}\right\}\times\left\{\sin\frac{2\pi+\theta}{p}.\sin\frac{2\pi-\theta}{p}\right\}\times\cdots\cdots$$

$$\cdots\cdots\left\{\sin\frac{\left(\frac{p}{2}-1\right)\pi-\theta}{p}.\sin\frac{\left(\frac{p}{2}-1\right)\pi+\theta}{p}\right\}\times\cos\frac{\theta}{p}$$

From the formula sin(A+B) sin(A–B) = $sin^2 A – cos^2$ B,

$$\sin\theta=2^{p-1}\sin\frac{\theta}{p}\left\{sin^2\frac{\pi}{p}-sin^2\frac{\theta}{p}\right\}\left\{sin^2\frac{2\pi}{p}-sin^2\frac{\theta}{p}\right\}\cdots\cdots$$

$$\cdots\cdots\left\{\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p}-\sin^2\frac{\theta}{p}\right\}\cos\frac{\theta}{p} \qquad (7)$$

Now, $=\dfrac{\sin\theta}{\sin\frac{\theta}{p}}=2^{p-1}\left\{sin^2\frac{\pi}{p}-sin^2\frac{\theta}{p}\right\}\left\{sin^2\frac{2\pi}{p}-sin^2\frac{\theta}{p}\right\}\cdots\cdots$

$$\cdots\cdots\left\{sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p}-sin^2\frac{\theta}{p}\right\}\cos\frac{\theta}{p} \qquad (8)$$

Suppose $\theta\to0$, then $\lim_{\theta\to\infty}\dfrac{\sin\theta}{\sin\frac{\theta}{p}}=\dfrac{\lim_{\theta\to\infty}\frac{\sin\theta}{\theta}}{\lim_{\theta\to\infty}\frac{\sin\frac{\theta}{p}}{\frac{\theta}{p}}}.\text{p}=\text{p},\lim_{\theta\to\infty}\cos\frac{\theta}{p}$

and $\theta \to 0 \sin^2 \frac{\theta}{p} = 0$. The limiting value of the equation (8) is given by

$$\mathrm{P} = 2^{p-1} sin^2 \frac{\pi}{p} sin^2 \frac{2\pi}{p} sin^2 \frac{3\pi}{p} \cdots\cdots sin^2 \frac{\left(\frac{p}{2}-1\right)\pi}{p} \tag{9}$$

Now dividing equation (7) by (8),

$$\sin\theta = \mathrm{p}\sin\frac{\theta}{p}\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{\pi}{p}}\right\}\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{2\pi}{p}}\right\}\cdots\cdots\left\{1-\frac{\sin^2\frac{\theta}{p}}{\sin^2\frac{\left(\frac{p}{2}-1\right)\pi}{p}}\right\}\cos\frac{\theta}{p} \tag{10}$$

Suppose p→∞, then $\lim_{\theta\to\infty}\left(psin\frac{\theta}{p}\right) = \left\{\lim_{\theta\to\infty}\frac{\sin\frac{\theta}{P}}{\sin\frac{\theta}{p}}\right\}.\theta = 1.\theta = \theta$,

and $\lim_{\theta\to\infty}\frac{sin^2\frac{\theta}{p}}{sin^2\frac{r\pi}{p}} = \frac{\left(\lim_{\theta\to\infty}\frac{sin\frac{\theta}{P}}{sin\frac{\theta}{p}}\right)^2}{\left(\lim_{\theta\to\infty}\frac{sin^2\frac{r\pi}{p}}{\frac{r\pi}{p}}\right)^2}.\frac{\theta^2}{r^2\pi^2} = \frac{1^2}{1^2}.\frac{\theta^2}{r^2\pi^2} = \frac{\theta^2}{r^2\pi^2}$

and $\lim_{\theta\to\infty}\cos\frac{\theta}{p}$

So $\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)$... ... ad.inf.

$$\Rightarrow \sin\theta = \theta\prod_{r\to1}^{r\to\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right). \tag{11}$$

Now for there solution of cos θ as an infinite product from Schlömilch's method (Trickovic, Stankovic & Vidanovic 2019).

$$\sin 2A = 2\sin A.\cos A = 2\sin\frac{A}{2}.\cos\frac{A}{2} = 2\sin\frac{A}{2}.\sin\left(\frac{\pi}{2}+\frac{A}{2}\right)$$

Then $\sin A = 2\sin\frac{A}{2}.\sin\left(\frac{\pi}{2}+\frac{A}{2}\right) = 2\sin\frac{A}{2}.\sin\left(\frac{\pi+A}{2}\right)$

$$\sin\theta = 2\sin\frac{\theta}{2}.\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}.\sin\left(\frac{\pi+\theta}{2}\right) \qquad (12)$$

Substituting $\theta = \frac{\theta}{2}$ in equation (12)

$$\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2^2}\sin\frac{\pi+\frac{\theta}{2}}{2} = 2\sin\frac{\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}$$

and $\sin\frac{\pi+\theta}{2} = 2\sin\frac{\pi+\theta}{2^2}\sin\frac{\pi+\frac{\pi+\theta}{2}}{2} = 2\sin\frac{\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2} = 2 = 2$.

Substituting these value in equation (12), after rearranging them

$$\sin\theta = 2^3\sin\frac{\theta}{2^2}\sin\frac{\pi+\theta}{2^2}\sin\frac{2\pi+\theta}{2^2} = \sin\frac{3\pi+\theta}{2^2}$$

Applying successively (12) for these four factors,

$$\sin\theta = 2^7\sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}\sin\frac{3\pi+\theta}{2^3}\sin\frac{4\pi+\theta}{2^3}\sin\frac{5\pi+\theta}{2^3}\sin\frac{6\pi+\theta}{2^3}\sin\frac{7\pi+\theta}{2^3}$$

$$= 2^{(2^3-1)}\sin\frac{\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\cdots\cdots\cdots\sin\frac{(2^3-1)\pi+\theta}{2^3} = \cdots\cdots\cdots$$

$$= 2^{(2^n-1)}\sin\frac{\theta}{2^n}\sin\frac{\pi+\theta}{2^n}\cdots\cdots\cdots\sin\frac{(2^n-2)\pi+\theta}{2^n}\sin\frac{(2^n-2)\pi+\theta}{2^n}$$

$$= 2^{(2^p-1)}\sin\frac{\theta}{p}\sin\frac{\pi+\theta}{p}\sin\frac{2\pi+\theta}{p}$$

$$\cdots\cdots\cdots\cdots\sin\frac{(p-2)\pi+\theta}{p}\sin\frac{(p-1)\pi+\theta}{p} \qquad (13)$$

Here P=2^n. Now substituting($\frac{\pi}{2}+\theta$) for θ in above equation,

$\cos\theta = 2^{p-1}\sin\frac{\pi+2\theta}{2p}\sin\frac{3\pi+2\theta}{2p}\cdots\cdots\sin\frac{(2p-3)\pi+2\theta}{2p}\sin\frac{(2p-1)\pi+2\theta}{2p}$. The last factor

$$= \sin\frac{(2p\pi)-(\pi-2\theta)}{2p} = \sin\left(\pi-\frac{\pi-2\theta}{2p}\right) = \sin\frac{\pi-2\theta}{2p}.$$

The last one factor is

$$=\sin\frac{(2p\pi)-(3\pi-2\theta)}{2p}=\sin\left(\pi-\frac{3\pi-2\theta}{2p}\right)=\sin\frac{3\pi-2\theta}{2p},\text{ and so on.}$$

Now combining the first and last factors, the second and the last but one factors and so on, and using the formula sin(A+B) sin(A+B)=sin^2A-sin^2B

$$\cos\theta=2^{p-1}\left\{sin^2\frac{\pi}{2p}-sin^2\frac{2\theta}{2p}\right\}\left\{sin^2\frac{3\pi}{2p}-sin^2\frac{2\theta}{2p}\right\} \tag{14}$$

Let $\theta \to 0$ then $\lim_{\theta\to0}\cos\theta = 1$ and $\lim_{\theta\to0}\sin^2\frac{2\theta}{2p} = 0$

$$1=2^{p-1}sin^2\frac{\pi}{2p}sin^2\frac{3\pi}{2p}sin^2\frac{5\pi}{2p} \tag{15}$$

Now dividing equation (14) by (15), we get

$$\cos\theta=\left\{1-\frac{sin^2\frac{2\theta}{2p}}{sin^2\frac{\pi}{2p}}\right\}\left\{1-\frac{sin^2\frac{2\theta}{2p}}{sin^2\frac{3\pi}{2p}}\right\}\left\{1-\frac{sin^2\frac{2\theta}{2p}}{sin^2\frac{5\pi}{2p}}\right\}\cdots\cdots$$

Let p⟶∞, then $\lim_{p\to\infty}\frac{sin^2\frac{2\theta}{2p}}{sin^2\frac{\pi}{2p}}=\frac{\left(\lim_{p\to\infty}\sin\frac{2\theta}{2p}\right)^2}{\left(\lim_{p\to\infty}\frac{\sin\frac{\pi}{2p}}{\frac{\pi}{2p}}\right)^2}.\frac{4\theta^2}{\pi^2}=\frac{1^2}{1^2}.\frac{4\theta^2}{\pi^2}=\frac{4\theta^2}{\pi^2}$, and so on.

$$\text{Hence }\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots\ldots\text{ad.inf.}$$

$$\Rightarrow\cos\theta=\prod_{r\to1}^{r\to\infty}\left(1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right)$$

RESULTS AND DISCUSSION

The convergence of the infinite products can be described as; if Σu_n be converted into infinitely small with same sign then (Asatryan 2015), $\Pi(1+u_n)$ converges absolutely, if $\Sigma|u_n|$ converges. If $\Pi(1+u_n)$ tends to have finite limiting value and $\Sigma|u_n|$ is divergent then its infinite product becomes semi-convergent. $\sin\theta$ as an infinite product is

$$\sin\theta = \theta \prod_{r\to 1}^{r\to\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right) \tag{16}$$

Since $\sum\left|-\frac{\theta^2}{r^2\pi^2}\right|$ i.e., $-\frac{\theta^2}{\pi^2}\sum\frac{1}{r^2}$ is convergent, so infinite product in equation (16) is convergent. Each quadratic factor in the infinite product in equation (xvi) can be resolved into linear factors in θ, i.e.,

$$\sin\theta = \theta\left(1+\frac{\theta}{\pi}\right)\left(1-\frac{\theta}{\pi}\right)\left(1+\frac{\theta}{2\pi}\right)\left(1-\frac{\theta}{2\pi}\right) \tag{17}$$

Since the series $\Sigma\,\frac{1}{r}$ is divergent, therefore the product in the equation (xvii) is semi-convergent.

$$\cos\theta = 11\left[1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right] \tag{18}$$

Equation (18) absolutely convergent, series $\Sigma\left|-\frac{4\theta^2}{2r-1)^2\pi^2}\right|$ *i.e.,* $\frac{4\theta^2}{\pi^2}\Sigma\frac{1}{(2r-1)^2}$ is convergent.

Each quadratic factor in (18) can be resolved into linear factors in θ,

i.e., $$\cos\theta = \left(1+\frac{2\theta}{\pi}\right)\left(1-\frac{2\theta}{\pi}\right)\left(1+\frac{2\theta}{3\pi}\right)\left(1-\frac{2\theta}{3\pi}\right) \tag{19}$$

The product in the form (19) is semi-convergent, as the series $\Sigma\frac{1}{2r-1}$ is divergent.

Hence, the infinite product (18) is absolutely convergent, while the infinite product (19) is semi-convergent (Asatryan 2015).

Some important Series from Infinite Product

To find a series for $\cot x$ for all values of x except zero or multiple of π.

From equation (11)

$$\sin\theta = \theta \prod_{r\to 1}^{r\to\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right)$$

$$\Rightarrow \sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\cdots\cdots \text{ad.inf.}$$

Taking logarithms of both sides,

$$\log\ \sin\theta = \log\theta + \log\left(1-\frac{\theta^2}{\pi^2}\right)\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\ldots\ldots \text{ad.inf.}$$

Differentiating both sides with respect to x, as the series on the R.H.S. admits of term by term differentiation, except when $\theta = 0$ or $\pm n\pi$,

$$\text{Now, } \frac{1}{\sin\theta}.\cos\theta = \frac{1}{\theta} + \frac{1}{1-\frac{\theta^2}{\pi^2}}\left(-\frac{2\theta}{\pi^2}\right) + \frac{1}{1-\frac{\theta^2}{2^2\pi^2}}\left(-\frac{2\theta}{2^2\pi^2}\right) + \ldots\ldots \text{to inf.}$$

$$\cot\theta = \frac{1}{\theta} + \frac{2\theta}{\theta^2-\pi^2} + \frac{2\theta}{\theta^2-2^2\pi^2} + \ldots\ldots \text{to inf.}$$

$$\cot\theta = \frac{1}{\theta} + \sum_{n=1}^{n=\infty}\frac{2\theta}{\theta^2-n^2\pi^2}.$$ Resolving into partial fraction

$$\cot\theta = \frac{1}{\theta} + \frac{(\theta-\pi)+(\theta+\pi)}{(\theta-\pi)(\theta+\pi)} + \frac{(\theta-2\pi)+(\theta+2\pi)}{(\theta-2\pi)(\theta+2\pi)} + \ldots\ldots \text{to inf.}$$ On solving,

to inf. Similarly other trigonometric series can be evaluated (Asatryan 2015).

CONCLUSION

In this paper,derivation of sine function and cosine function as an infinite product by Schlömilch's method, has been shown.

Infinite products of form $-\frac{\theta^2}{r^2}\ \Sigma\frac{1}{r^2}$ are convergent; while quadratic factors of form $\Sigma\frac{1}{r^2}$ are divergent that makes it semi-convergent. Absolute convergent series are also convergent. Semi-convergent of the form $\Sigma\frac{1}{2r-1}$ are divergent. Some important series of infinite product has been also discussed.

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