GROUP THEORETIC STUDY OF BRENKE TYPE POLYNOMIALS

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INTRODUCTION

The Brenke type polynomials $G_n^{(\alpha,p)}(x)$ defined by
where α ($\alpha \neq -1, -2, -3, \ldots$), p and n are independent of each other, satisfies the

$$G_n^{(\alpha,p)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n \, (p)_k \, x^k}{(1+\alpha)_k \, k! \, (n-k)!}$$

differential equation

$$x(1+x)\frac{d^2V}{dx^2} + \{\alpha+1+(1-n+p)x\frac{dV}{dx} - npV = 0$$

Generating functions of this polynomial in various forms have been obtained by the authoress [2]. (Bajracharya; 1995)

The purpose of this paper is to derive some generating functions for this polynomial through the L.weisner's group theoretic method [3] which consists in contructing a partial differential equation from the ordinary differential equation by giving suitable interpretation to the index n to derive the elements of lie algebra and the corresponding Lie group which is admitted by the partial differential equation. The main results of the paper are listed below:

$$(1-t)^n G_n^{(\alpha,p)}\left(\frac{x}{1-t}\right) = \sum_{r=0}^{n} \frac{(-\alpha-n)_r}{r!} G_{n-r}^{(\alpha,p)}(x)\, t^r$$

$$(1-z)^{-c} \; {}_2F_1\left[c; 1+\alpha; \frac{-xz}{z-1}\right] = \sum_{r=0}^{\infty} \frac{(c)_r}{(1+\alpha)_r} G_r^{(\alpha,p)}(x) z^r$$

$$(1-z)^{-\alpha-1+p-n} \{1-z(1+x)\}^{-p} G_n^{(\alpha,p)} \left(\frac{x}{1-z(1+x)}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(n+1)}{r!} G_{n+r}^{(\alpha,p)}(x) \; z^r$$

$$(1-z)^{-\alpha-1}\left(1-\frac{zx}{1-z}\right)^{-p} = \sum_{r=0}^{\infty} G_r^{(\alpha,p)}(x) \; z^r.$$

$$(1-z)^{-\alpha-1-p} \{1-z(1+x)\}^{-p} \left(\frac{z(w-1)+1}{(z-1)w}\right)^n G_n^{(\alpha,p)} \left(\frac{xz}{\left(2z+\frac{1}{w}\right)\{1-z(1+x)\}}\right)$$

$$= \sum_{s=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^r(-\alpha-n)_r(n-r+1)_s}{r!s!} G_{n-r+s}^{(\alpha,p)}(x) \; z^{n-r+s}$$

$$e^z \; {}_1F_1(p; 1+\alpha; xz) = \sum_{n=0}^{\infty} \frac{G_n^{(\alpha,p)}(x) \; z^n}{(1+\alpha)_n}$$

$$(1-z)^{-\alpha-1+p} \{1-z(1+x)\}^{-p} e^{\frac{-wz}{1-z}} \; {}_1F_1\left[p; 1+\alpha; \frac{-wxz}{\{1-z(1+x)\}(1-z)}\right]$$

$$= \sum_{n=0}^{\infty} \frac{n! \; L_n^{(\alpha)}(x)}{(1+\alpha)_n} G_n^{(\alpha,p)}(x) \; z^n$$

$$e^t \; {}_1F_1\left[p;\, 1+\alpha;\, \frac{-xt}{1+x}\right] = \sum_{n=0}^{\infty} G_n^{(\alpha,p)}\left(\frac{-x}{1+x}\right) t^n$$

GROUP OF OPERATORS

We Replace d/dx by $\partial/\partial x$, n by $z\partial/\partial z$ and v by v(x,z) in (1.2), we arrive at the partial differential equation

$$x(1+x)\frac{\partial^2 v}{\partial x^2} - zx\,\frac{\partial^2 v}{\partial z\partial x} + (\alpha+1+x+px)\frac{\partial v}{\partial x} - pz\,\frac{\partial v}{\partial z} = 0$$

Since $v = G_n^{(\alpha,p)}(x)$ is a solution (1.2), $v_1 = G_n^{(\alpha,p)}(x)\, z^n$ is a solution of the equation $Lv = 0$, where

$$L = x(x+1)\,\frac{\partial^2}{\partial x} - zx\frac{\partial^2}{\partial z\partial x} + (\alpha+1+x+px)\frac{\partial}{\partial x} - pz\frac{\partial}{\partial z}.$$

Let us use two first order linear differential operators B and C defined by

$$B = \frac{x}{z}\frac{\partial}{\partial x} - \frac{\partial}{\partial z}$$

and

$$C = zx(1+x)\frac{\partial}{\partial x} + z^2\frac{\partial}{\partial z} + z(\alpha+px+1).$$

such that

$$B\left[G_n^{(\alpha,p)}(x)\, z^n\right] = -(\alpha+n)\, G_{n-1}^{(\alpha,p)}(x)\, z^{n+1}$$

$$C\left[G_n^{(\alpha,p)}(x)\, z^n\right] = (n+1)\, G_{n+1}^{(\alpha,p)}(x)\, z^{n+1}$$

To find the Lie group of operators, let us write $A = z\,\partial/\partial z$ then we have the following commutator relations

$[A,B] = -B,\ [A, C] = C,\ [B, C] = -(2A + \alpha + 1)$

where $[A,B] = AB - BA.$

These commutator relations show that the operators 1,A, B, C generate a three parameter Lie group . Using (2.4) we can write

$xL = CB + A^2 + \alpha A$

From (2.4) and (2.5), we find that

$[xL,C] = [xL,B] = [xL,C] = 0.$

which shows that each of these operators A,B,C commute with xL.

The extended forms of the group generated by B and C are given by

$$e^{bB} f(x,z) = f\left(\frac{xz}{z-b}, z-b\right)$$

andwhere b and c are arbitrary numbers.From (2.7) and (2.8) it follows that

$$e^{cC} f(x,z) = (1-cz)^{-\alpha-1+p}\ \{1-cz(1+x)\}^{-p}\ f\left(\frac{x}{1-cz(1+x)}, \frac{z}{1-cz}\right)$$

From (2.7) and (2.8) it follows that

$$e^{cC}\, e^{bB}\, f(x,z) \;=\; (1-cz)^{-\alpha-1+p}\left\{\, 1-cz(1+x)\right\}^{-p}$$

$$F\left(\frac{xz}{(z-b+cz)1-cz(1+x)},\ \frac{z}{1-cz}\ -\ b\right).$$

CONJUGATE SETS

$$e^{\alpha A}\, B\, e^{-\alpha A} \;=\; B\, e^{-\alpha}$$

$$e^{\alpha A}\, C\, e^{-\alpha A} \;=\; C\, e^{\alpha}$$

$$e^{bB}\, C\, e^{-bB} \;=\; -2bA\ -\ b^2B\ +\ C\ -b\,(\alpha+\ 1)$$

$$e^{cC}\, A\, e^{-cC} \;=\; 2cA\ +\ B\ -\ c^2C\ +\ c(\alpha\ +\ 1).$$

$$e^{bB}\, C\, e^{-bB} \;=\; -\ 2bA\ -\ b^2\ +\ C\ -\ b\,(\alpha+\ 1).$$

We attempt to determine the functions annulled by L and $R = r_1A + r_2B + r_3C + r_4$ where the r's are arbitrary constants other than $r_1 = r_2 = r_3 = r_4 = 0$. Since the operators commute with xL it is sufficient to consider one operator from each of the conjugate sets into which the operator R fall with respect to the group G. To find the conjugate sets of operators of the first order for this polynomial we take $S = e^{cC}e^{bB}$. Then for each choice of b and c, $S(A-n)S^{-1}$ represents an operator conjugate to A-n. We compute the following conjugate operators:

Then for $S = e^{cC}\, e^{bB}$, we have

$$SAS^{-1} = e^{cC} e^{bB} A e^{-bB} e^{-cC}$$

$$= e^{cC} (A + bB) e^{-cC}$$

$$= A - cC + b[2cA + B - c^2 C + c(\alpha + 1)]$$

$$= (1 + 2bc)A + bB - c(1+bc)C + bc(1 + \alpha).$$

Therefore for $R = r_1A + r_2B + r_3C + r_4$, (A-n) is conjugate to R if $r_1 = 1 + 2bc$, $r_2 = b$, $r_3 = -c(1 + bc)$. From two of these equations, we can find b and c in terms of r_1, r_2, and r_3. The third equation then imposes a restrictive relation on r_i (i = 1, 2, 3). If $r_2 = 0$ then b = 0, c = $-r_3$ and $r_1 = 1$, if $r_2 \neq 0$ then

$$r_2 = b,\ c = \frac{rr_1 - 1}{2r_2} \text{ and } r_3 = \frac{-(r_1 - 1)}{2r_2}\left(1 + \frac{r_1 - 1}{2}\right), \text{i.e.}$$

$r_1 2 + 4r_2r_3 = 1$. In other words, for all possible choices of b and c, $r_1 2 + 4r_2r_3 \neq 0$. Therefore (A-n) is not conjugate to the set of operators for $r_1 2 + 4r_2r_3 = 0$.

GENERATING FUNCTIONS DERIVED FROM THE FIRST ORDER OPERATOR (A-n)

We know that $v_1(x,z) = G_n^{(\alpha,p)}(x)\, z^n$ is annulled by L and A-n where $A = z\ \partial/\partial z$. We now transform $v_1(x,z)$ by the operator $e^{cC}\ e^{bB}$ and then consider the following three different cases:

Case 1. b = 1,c = 0. In this case we have

$$e^B \left[G_n^{(\alpha,p)}(x) z^n \right] = G_n^{(\alpha,p)}\left(\frac{xz}{z-1}\right)(z-1)^n$$

We use (2.3) and then replace z^{-1} by t to arrive at

$$(1-t)^n G_n^{(\alpha,p)}\left(\frac{x}{1-t}\right)=\sum_{r=0}^{n}\frac{(-\alpha-n)_r\, G_{n-r}^{(\alpha,p)}(x)\,t^r}{r!}$$

Further by omitting the constant factor , and replacing -n by c, the right side of (4.1) can be written as

$$(1-z)^{-c}\ {}_2f_1\left(c,p;1+\alpha;\frac{-xz}{z-1}\right)=\sum_{r=0}^{\infty}\frac{(c)_r}{(1+\alpha)_r}G_r^{(\alpha,p)}(x)\,z^r$$

Case 2. b = 0 , c = 1. For these values, we have.

$$e^C\left[z^n G_n^{(\alpha,p)}(x)\right]$$

$$=(1-z)^{-\alpha-1+p}\{1-z(1+x)\}^{-p}\left(\frac{z}{1-z}\right)^{-n}G_n^{(\alpha,p)}\left(\frac{x}{1-z(1+x)}\right)$$

On the other hand we find that

$$e^C\left[z^n\ G_n^{(\alpha,p)}(x)\right]\ =\ \sum_{r=0}^{\infty}\frac{(n+1)_r}{r!}\ G_{n+r}^{(\alpha,p)}(x)\,z^{n+r}$$

Equating (4.4) and (4.5) and then dividing by z^n we get

$$(1-z)^{-\alpha-1+p-n}\ \{1-z(1+x)\}^{-p}\ G_n^{(\alpha,p)}\left(\frac{x}{1-z(1+x)}\right)$$

$$=\ \sum_{r=0}^{\infty}\frac{(n+1)_r}{r!}\ G_{n+r}^{(\alpha,p)}(x)\,z^r$$

For n = 0, we have

$$(1-z)^{-\alpha-1}\left(1-\frac{zx}{1-z}\right)^{-p} = \sum_{r=o}^{\infty} G_r^{(\alpha,p)}(x)\, z^r$$

which is a familiar generating function for the polynomial $G_n^{(\alpha,p)}(x)$ deduced earlier by the authoress [2]

Case 3. **bc ≠ 0** . We choose c = 1 and b = -1/w so that for all finite values of w. bc ≠ 0. In this case we have

$$e^C e^{-\frac{1}{w}} B\left\{z^n G_n^{(\alpha,p)}(x)\right\} = (1-z)^{-\alpha-1+p}\left\{1-z(1+x)\right\}^{-p}\left(\frac{z(w-1)+1}{(1-z)w}\right)$$

$$G_n^{(\alpha,p)}\left(\frac{xz}{(2z+1/w)(1-z-zx)}\right)$$

The right side of this can be writtern in the form

$$e^C e^{(-1/w)} B\left[z^n G_n^{(\alpha,p)}(x)\right] = \sum_{r=o}^{n} \frac{(-1)^r}{w^r}\cdot\frac{1}{r!}\,\frac{1}{s!}\,(-\alpha-n)_s$$
$$\cdot(n-r+1)_s\, G_{n-r+s}^{(\alpha,p)}(x)\, z^{n-r+s}$$

Equating (4.8) and (4.9), we arrive at a new generating relation

$$(1-z)^{-\alpha-1+p-n}\{1-z(1+x)\}^{-p}\{z(w-1)+1\}^n w^{-n}\; G_n^{(\alpha,p)}\left(\frac{xz}{(2z+\frac{1}{w})(1-z-zx)}\right)$$

$$=\sum_{s=0}^{\infty}\sum_{r=0}^{n}(-1)^r\frac{1}{w^r}\frac{1}{r!}\frac{1}{s!}(-\alpha-n)_r\,(n-r+1)_s\; G_{n-r+s}^{(\alpha,p)}(x)\, z^{n-r+s}$$

GENERATING FUNCTIONS DERIVED FROM OPERATORS NOT CONJUGATE TO (A-n)

The operators for which $r_1 2 + 4r_2r_3 = 0$ may be considered under the following three cases;

Case 1. $r_1 = 0, r_2 = 1, r_3 = 0$,:. Now we will solve the system Lu = 0 and $(B+\eta)u = 0$ where η is a non-zero constant. For convenience, we choose $\eta = 1$ and write the equations

$$x(1+x)\frac{\partial^2 u}{\partial x^2} - zx\frac{\partial^2 u}{\partial z \partial x} + (a+1+x+px)\frac{\partial u}{\partial x} - pz\frac{\partial u}{\partial z} = 0$$

$$xz^{-1}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial z} + u = 0$$

and

Solving (5.1) and (5.2) , we find that one solution of this is

$$u(x,z) = e^z \,{}_1F_1(p;\ 1+\alpha;\ xz).$$

The function u(x,z) can be expanded in the form

$$e^z \,{}_1F_1(p;\ 1+\alpha;\ xz) = \sum_{n=0}^{\infty} \frac{G_n^{(\alpha,p)}(x)\, t^n}{(1+\alpha)_n}$$

Hence u(x,z) represents a generating function for the polynomial $G_n^{(\alpha,p)}(x)$

Case 2. $r_1 = 2$, $r_2 = 1$, $r_3 = -1$. In this case, we seek a solution of the system Lu = 0 and $(2A + B - C + \lambda)u = 0$ where λ is a non-zero constant. To solve this system, we consider the fact that

$$e^C B e^{-C} = 2A + B - C + \alpha + 1$$

and

$$e^{C}(B - u)\, e^{-C} = 2A + B - C + (\alpha + 1 - u).$$

Let $S_1 = e^{C}$, then we have, $S_1 (B - w) S_1^{-1} (S_1 (u)) = 0$ if $(B\text{-}w)u = 0$.
Therefore u is annulled by L and $2A + B - C + 1 + \alpha - w$.

From ξ 5., **Case 1,** when $\eta = -w$, we obtain

$$S_1\, u(x,-wz) = (1-z)^{-\alpha-1+p} (1-z-zx)^{-p}\, e^{\frac{-wz}{(1-z)}}$$

$$ {}_1F_1\left[p; 1+\alpha; -\frac{wxz}{(1-z-zx)(1-z)}\right]$$

$$= {}_2F_1[-n, p; 1+\alpha; -x]\; L_n^{(\alpha)}(w)\, z^n$$

$$= \sum_{n=0}^{\infty} \frac{n!\, L_n^{(\alpha)}(w)}{(1+\alpha)_n}\, G_n^{(\alpha,p)}(x)\, z^n$$

Similar results have been deduced earlier by Abdul-Halim & Al-Salam [1],Rainville [4], Weisner [5] in different methods. (Abdul, 1963; Rainville, 1960; Weisner, 1955).

Thus we have derived a new bilateral generating function involving Laguerre polynomial for the polynomial $G_n^{(\alpha,p)}(x)$ in the form.

$u(x,-wz) = e^{-wz}\, {}_1F_1(p; 1+\alpha; -wxz)$
as a solution of the system $Lu = 0$ and $B - w = 0$ .Further we find that

$$(1-z)^{-\alpha-1+p}(1-z-zx)^{-p}\, e^{\frac{-wz}{(1-z)}}\ {}_1F_1\left[p;1+\alpha;-\frac{wxz}{(1-z-zx)(1-z)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{n!\,L_n^{(\alpha)}(w)}{(1+\alpha)_n}\,G_n^{(\alpha,p)}(x)\,z^n$$

Case 3. $r_1 = 0,\ r_2 = 0,\ r_3 = 1.$ Next we seek a solution of the system Lu=0 and $(C+\lambda)u = 0$ where λ is a non-zero constant. To solve this system, we make use of the solution which we have already obtained in Case 1. i.e, the system Lu=0 and $(B+\eta)u = 0$. Now let us find b and c such that

$$e^{bB}e^{cC}B\,e^{-cC}e^{-bB} = kC$$

where k is a non-zero constant. This can be written as

$$e^{bB}e^{cC}B\,e^{-cC}e^{-bB}$$

$$= e^{bB}\,[\,2cA + B - c^2C + c\,(\alpha+1)\,]\,e^{-bB}$$

$$= 2c(A+bB) + B - c^2\{-2bA - b^2B + C + b(-\alpha-1) + c(\alpha+1)\}$$

$$= 2c(1+bc)A + (1+bc)^2B - c^2C + (\alpha+1)c(bc+1)$$

If we choose $b = 1$, $c = -1$ and let $S_2 = e^{B}e^{-C}$ then we arrive at

$$S_2 B\, S_2^{-1} = -C$$

and hence

$$S_2\,(B+\eta)\,S_2^{-1} = -C + \eta.$$

If $(B+\eta)u = 0$, then $S_2(B+\eta)\,S_2^{-1}(S_2(u)) = 0$. Therefore if u(x,z) is annulled by L and $B+\eta$, then $S_2u(x,z)$ is annulled by L and $C-\eta$. For $\eta = 1$, we have

from ξ 5. Case 1;

$$u(x,z) = e^{z}\ {}_1F_1[p;\,1+\alpha;\,xz]$$

The generating function, we now seek is a transformation of u(x,z) given by

$$S_2u(x,z) = e^{B}e^{-C}\,u(x,z).$$

$$S_2u(x,z) = e^{B}\,(1+z)^{-\alpha-1+p}\,(1+z+zx)^{-p}\,e^{\frac{z}{1+z}}$$

$$\,{}_1F_1\left[p;\,1+\alpha;\,\frac{xz}{(1+z+zx)(1+z)}\right]$$

$$= z^{-\alpha-1+p}\,z^{-p}\,(1+x)^{-p}\,e^{\frac{z-1}{z}}\ {}_1F_1\left[p;1+\alpha;\,\frac{x}{z(1+x)}\right]$$

$$= e^{-\alpha-1}\,(1+x)^{-p}\,e^{-\frac{1}{z}}\ {}_1F_1\left[p;1+\alpha;\,\frac{x}{z(1+x)}\right]$$

We employ the relations (2.7) and (2.8) to write in the form
Putting z = -1/t, the right member of (5.10) may be written in the form

$$e(-t)^{\alpha+1}\,(1+x)^{-p}\,e^{t}\ {}_1F_1\left[p;1+\alpha;\,\frac{x}{z(1+x)}\right].$$

Since

$$e^{t}\ {}_1F_1\left[p;1+\alpha;\,\frac{-xt}{1+x}\right] = \sum_{n=0}^{\infty}\frac{G_n^{(\alpha,p)}\left(\frac{-x}{1+x}\right)}{(1+\alpha)_n}\,t^{n}$$

we may consider (5.11) as a generating function for the polynomial $G_n^{(\alpha,p)}(x)$ where $z = -x/(1+x)$.

Again if we restrict α to be a positive integer then we can write the right member of (5.10) in the form

$$e(-1)^{\alpha+1}(1+x)^{-p} \sum_{n=1+\alpha}^{\infty} \frac{G_{n=\alpha-1}^{(\alpha,p)}\left(\dfrac{-x}{(1+x)}\right)}{(1+\alpha)_{n-\alpha-1}} t^n$$

Hence we have established all desired results.

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