APPLICATION OF DIFFERENTIAL EQUATION TO POPULATION GROWTH

Harideo Chaudhary*

ABSTRACT

Thomas Malthus, an 18th century English scholar, observed an essay written in 1798 that the growth of the human population is fundamentally different from the growth of the food supply to feed that population. He wrote that the human population was growing geometrically [i.e. exponentially] while the food supply was growing arithmetically [i.e. linearly]. He concluded that left unchecked, it would only be a matter of time before the world's population would be too large to feed itself. The first growth model we examine in this module is the one Thomas Malthus referred to in his famous essay. Malthus' model is considered a more sophisticated model for the special case of world population.

Key words: *population, equation, differential, growth and logistic.*

INTRODUCTION

Malthus' model is commonly called the natural growth model or exponential growth model. For this we assume that the population grows at a rate that is proportional to itself (Banks, 1999, p.138). If *P* represents such population, then the assumption of natural growth can be written symbolically as;

dP/dt = k P

Where, *k* is a positive constant.

This model has many applications besides population growth. For example; the balance in a savings account with interest compounded continuously (and no withdrawals) exhibits natural growth. In this case, the constant *k* is called the annual rate of interest. Also, large animal populations whose size is not constrained by environmental factors grow exponentially. In this setting, *k* is called the productivity rate of the population.

NAÏVE MODEL: EXPONENTIAL GROWTH

It is possible to explain the various growth phenomena with mathematical model, some of them are simple and some are complicated. The most famous example is the familiar Malthusian or exponential growth model; in differential equation form it has the equation

$$\frac{dP}{dt}=KP$$

Where, *P* is the magnitude of growing quantity, *t* is the time and *k* is the growth coefficient (Banks, 1999, p.139). The solution to this equation is

$$P=P_0e^{-at}$$

* Mr. Chaudhary is an Associate Professor, Department of Engineering Science and Humanities, Pulchowk Campus, IOE, T.U., Lalitpur, Nepal.

Where, P_0 is the value of P when time $t = 0$. This can be written in standard form as

$$\frac{1}{P}\frac{dP}{dt} = K\ (t)$$

While the exponential model is useful for short term forecast, it gives unrealistic estimates for long time period. After just a few decades, population will rapidly grow toward infinity in this model. A more realistic model should capture the idea that population does not grow forever, but instead of level off around some long-term level. This leads us to our second model.

DERIVING THE LOGISTIC POPULATION GROWTH MODEL

About two hundred years ago, an English clergyman- economist named Thomas Malthus published a series of essays (1798, reprinted 1970) in which he contended that population grow according to the law of geometric progression. In other word, if a population of a country has a certain magnitude at a particular moment, then that population will double itself at the end of a specified time period and this periodic doubling of population will continue indefinitely.

Let us begin with a simple example. An educational institute has 10,000 people. At time zero, one person has a joke or a nice item suitable for gossip. The person tells the joke or gossip item to another person, so now two people know. These two each tell two more people and hence now there are four knowledgeable people. It is coming to our geometric progression. At any given moment people have heard the joke or gossip and are spreading it, these are the "infective" (Banks, 2001, p.29). At the same moment $P_* - P$ people have not heard the joke or gossip; these are "susceptible" (Banks, 2001, p.29). The quantity P_* is the total number of people in the community, in this case $P_* = 10,000$.

After a certain period of time, infective start telling the joke to those who are also infective. They are already heard the joke and are also spreading it around. After an additional period of time, susceptible are increasingly hard to find. That is everyone has heard the joke. If we were to carry out a so-called stochastic analysis of this problem we would obtain a "difference equation" that would tell us how many infective are at any time t. However if the assumption is made that the total population P_* is very large we can replace the stochastic analysis with a deterministic analysis and acquire a "differential equation". In this case we obtain the relationship

$$\frac{dP}{dt} = KP - bP^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

This is the equation for logistic growth

Here P is the number of infective (those who have already heard the joke), k is the growth coefficient, b is the crowding coefficient and bP^2 is the "breaking term" that prevents unlimited growth (Kreyszig, 1999, p.13). The number of susceptible (those who have not yet heard) is of course $P_* - P$. Defining the crowding coefficient as $b = k/P_*$ equation (1) becomes

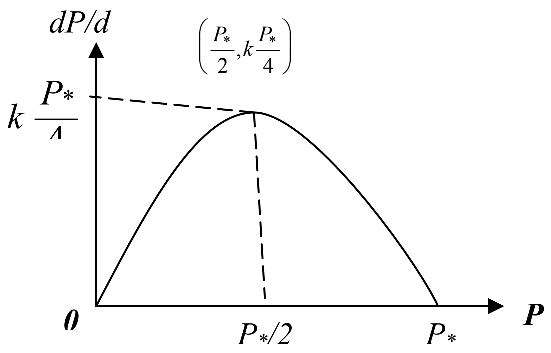
$$\frac{dP}{dt} = KP\ (1 - P/P_*) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

This is Verhulst's famous differential equation for **logistic growth** (Banks, 2001, p.30).

It can also be obtained by multiplying the exponential model by a factor ($1-P/P_*$). Here P_* is assumed a maximum long-term population a city can sustain. This is also known as so called equilibrium value or carrying capacity (Banks, 2001, p.25). In this model, the population starts out growing exponentially. But as P approaches the maximum level P_*, *the* term ($1-P/P_*$) approaches zero and slowing down the growth rate. This is known as "logistic model".

GEOMETRICAL INTERPRETATION OF LOGISTIC GROWTH FUNCTION

The graph of $\frac{dP}{dt}$ against P, where $\frac{dP}{dt}$ is given by equation (2) gives the graph of logistic growth function, which is a parabola with intercepts at (0, 0) and (P_*, 0) and with vertex at $\left(\frac{P_*}{2}, k\frac{P_*}{4}\right)$.



(a) When $0< P< P_*$, $\frac{dP}{dt}>0$, therefore P increases towards P_*.

(b) When $P> P_*$, $\frac{dP}{dt}<0$, therefore P decreases towards P_*.

Hence, we conclude that the population level $P(t)$ always approaches P_* which can be expressed as;

$$\lim_{t\to\infty} P(t) = P_*, \text{ provided } P_0>0.$$

(c) If $P=0$ or $P=P_*$, then $\frac{dP}{dt}=0$ and $P(t)$ does not change. The constant solution $P=0$ and $P=P_*$ are known as equilibrium solution (Pundir, 2006, p.71). Corresponding to equilibrium solutions, the points $P=0$ and $P=P_*$ are called equilibrium points or critical points.

SOLUTION OF LOGISTIC EQUATION

Let us consider the equation (2),

$$\frac{dP}{dt} = KP\,(1-P/P_*)$$

With initial condition, $P(0) = P_0$ ………………………………(3)

Equation (2) can be written as

$$P_* \frac{dP}{dt} = kPP_* - kP^2 = kP(P_* - P)$$

On separating the variables, we have

$$\frac{P_* dP}{P(P_* - P)} = K\,dt$$

Which can also be written as

$$\left[\frac{1}{P} + \frac{!}{P_* - P}\right] dP = K\,dt \quad \text{………………….... (4)}$$

On integrating (4), we get

logP-log (P-P) = kt+c, c* being constant of integration.

$$\Rightarrow \log \frac{P}{P_* - P} = kt + c \text{………………………………….... (5)}$$

Using (3), we get, $C = \log \frac{P_0}{P_* - P_0}$

$$\log \frac{P}{P_* - P} = \log e^{kt} + \log \frac{P_0}{P_* - P_0} = \log \frac{P_0 e^{at}}{P_* - P_0}$$

Which gives

$$\frac{P}{P_* - P} = \frac{P_0 e^{kt}}{P_* - P_0} \Rightarrow [(P_* - P_0) + P_0 e^{kt}]P = P_* P_0 \qquad P = \frac{P_*}{1 + \left[\frac{P_* - P_0}{P_0}\right] e^{-kt}}$$

$$\text{Hence, } P(t) = \frac{P_*}{1 + c_1 e^{-kt}} \text{ ……………………………………(6)}$$

Where, $c_1 = \frac{P_* - P_0}{P_0}$, a constant (Pundir, 2006, p.72).

GEOMETRICAL INTERPRETATION

Equation (6) represents the size of the population at any time *t*. From (6) it is also clear that $P(t) \to P_*$ as $t \to \infty$. Therefore, a population that satisfies the logistic equation is not like a naturally growing population, it does not grow without bound, but approaches the finite limiting population P_* *as* $t \to \infty$. *But in* this case since $\frac{dp}{dt} > 0$ therefore, population is steadily increasing (Pundir, 2006, p.73).

Now, differentiating (2) w.r.t. *t*, we have,

$$\frac{d^2 p}{dt^2} = k\left[\frac{dP}{dt} = \frac{2P}{P_*}\frac{dP}{dt}\right] = \frac{k}{P_*}\left[P_* - 2P\right]\frac{dP}{dt}$$

$$= \frac{k^2}{P_*^2} P(P_* - P)(P_* - 2P) \quad \text{… ……………………………………(7)}$$

Now we shall discuss the following cases:

(a) If $P_* -2P>0$, we get $P_* -P > P > 0$. Then, $\frac{d}{dt}\left[\frac{dP}{dt}\right]>0$, therefore, the rate of increase $\frac{dP}{dt}$ increases with time. Hence there is an accelerated growth of the population in the range $0 < P < P_*/2$.

(b) If $P_*/2 < P < P_*$, then $P_*- 2P<0$ and $P_* -P > 0$, therefore, $\frac{dP}{dt}$ is a decreasing function of time. Hence, there is a retarded growth of the population in range $P_*/2<P<P_*$ *(Pundir, 2006, p.73).*

BEHAVIOUR OF LOGISTIC CURVE

Let b be the birth rate and d be the death rate with the population size p. Let us assume that b decreases and d increases with P. Then we can write

$b = b_1-b_2p$, $d=d_1-d_2p$, $b_1,b_2,d_1,d_2 > 0$

Now,

$$\frac{dp}{dt} = [(b_1 - d_1) - (b_2 + d_2)p] = p\,[a - bp], \quad a>0,\ b>0 \quad \text{Where, } b-d=a \quad \ldots\ldots (8)$$

Separating the variables, we get

$$\frac{dp}{p[a-bp]} = dt \qquad \Rightarrow \left[\frac{b}{a}\frac{1}{(a-bp)} + \frac{1}{a}\frac{1}{p}\right]dp = dt$$

Integrating, we get,

$$\frac{b}{a}\log\frac{(a-bp)}{-b} + \frac{1}{a}\log p = t + \log c$$

$$\Rightarrow \frac{1}{a}[\log p - \log(a-bp)] = t + \log c$$

$$\Rightarrow \log\frac{p}{a-bp} - a\log c = ta \qquad \text{Let, } a\log c = \log c_1$$

Then, $\log\frac{p}{a-bp} - \log C_1 = at \quad \Rightarrow \quad \frac{p}{C_1(a-bp)} = e^{at} \quad \Rightarrow \quad p\ (t) = e^{at}[C_1(a-bp)]$

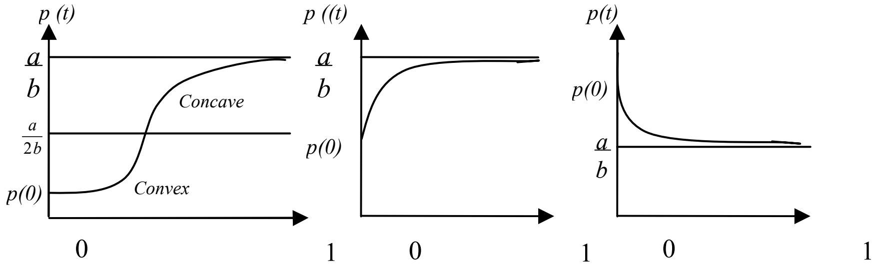
At t = 0, $p\ (0) = [C_1(a-bp(0)] \quad \Rightarrow \quad C_1 = \frac{p(0)}{a-bp} = C_1e^{at}$

Hence, $\frac{p}{a-bp} = C_1e^{at} \quad \Rightarrow \quad \frac{p(t)}{a-bp(t)} = \frac{p(0)}{a-bp(0)}e^{at}$

From (8), we have, $\frac{d^2p}{dt^2} = a - 2bp$

We can conclude that, $\frac{d^2p}{dt^2}$ *>=< 0 according as p>=<* $\frac{a}{2b}$

GRAPHICAL REPRESENTATION



CONCLUSION

(a) If $p\ (0) < \frac{a}{2b}$, $p\ (t)$ increases at an increasing rate till $p\ (t)$ reaches $\frac{a}{2b}$, and then it increases at a decreasing rate and approaches $\frac{a}{b}$ at $t \to \infty$, Hence, in this case, the growth curve is convex if $p < \frac{a}{2b}$ and concave if $p > \frac{a}{2b}$ and it has point of inflexion at $p = \frac{a}{2b}$,

(b) If $\frac{a}{2b}$ <p (0)< $\frac{a}{b}$, p (t) increases at a decreasing rate and approaches $\frac{a}{b}$ as at $t \to \infty$,

(c) If p (0) = $\frac{a}{b}$, then p (t) is always equal to $\frac{a}{b}$.

(d) If $p\ (0) > \frac{a}{b}$, $p\ (t)$ decreases at a decreasing rate and approaches $\frac{a}{b}$ as at $t \to \infty$.

WORKS CITED

Banks, R.B. (1999). *Slicing Pizzas, Racing Turtles, and Further Architectures in Applied Mathematics.* Princeton University Press..

Banks, Robert B. (2001). *Towing Icebergs, Falling Dominoes, and Other Architectures in Applied Mathematics.* Princeton University Press.

Kreyszig, Erwin (1999). *Advanced Engineering Mathematics.* John Willey and Sons Inc.

Pundir, S.K. and Rimple (2006). *Biomathematics.* Pragati Prakashan.