THEORETICAL DESCRIPTION OF A REPARAMATRIZATION OF DISCRETE TWO PARAMETER POISSON LINDLEY DISTRIBUTION FOR MODELING WAITING AND SURVIVAL TIMES DATA

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ABSTRACT

In this research paper, the theoretical description of a reparamatrization of a discrete two-parameter Poisson Lindley Distribution, of which Sankaran's (1970) one parameter Poisson Lindley Distribution is a special case, is derived by compounding a Poisson Distribution with two parameters Lindley Distribution for modeling waiting and survival times data of Shanker et al. (2012).The first four moments of this distribution have derived. Estimation of the parameters by using method of moments and maximum likely hood method has been discussed.

Key Words: Compounding, reparamatrization, moments, estimation of parameters, maximum likely hood, probability generating function, moment generating function, two-parameter Lindley distribution.

INTRODUCTION

Lindley (1958) introduced a one parameter Lindley distribution, given by its probability density function

$$f(x; \theta) = \frac{\theta^2}{\theta+1}(1 + x)e^{-\theta x}, x > 0, \theta > 0 (1.1)$$

Similarly one parameter Poisson Lindley distribution (PLD) given by its probability mass function as

$$f(x; \theta) = \frac{\theta^2(x+\theta+2)}{(\theta+1)^{x+3}}, x = 0,1,2,\ldots; \ \theta > 0 (1.2)$$

This distribution has been introduced by Sankaran (1970) to model count data.

The distribution arises from a Poisson distribution when its parameter follows a Lindley distribution (1.1).

There paramatrization one parameter PLD is given by probability mass function as

$$f(x; \theta) = (1 + 2\theta + \theta x)\frac{\theta^x}{(1+\theta)^{x+3}}, x = 0,1,2,\ldots; \ \theta > 0 (1.3)$$

Recently, Shanker *et al.* (2012) obtained a two parameter Lindley distribution given by the probability density function

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$$f(x; \alpha, \theta) = \frac{\theta^2}{\theta + \alpha}(1 + \alpha x)e^{-\theta x}, x > 0, \theta > 0, \alpha > 0 (1.4)$$

For α = 1, the distribution reduces to the one parameter Poisson Lindley distribution. This distribution has been found to be a better model then one parameter PLD for analyzing waiting and survival times and grouped mortality data.

Suppose that the parameter λ of a Poisson distribution follows the two parameter LD (1.4). Then the two parameter Lindley mixture of Poisson distribution becomes

$$f(x; \lambda, \theta) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{\Gamma(1+x)} \cdot \frac{\theta^2}{\theta + \alpha}(1 + \alpha\lambda)e^{-\theta\lambda}d\lambda, \ x > 0, \alpha > 0, \lambda > 0, \theta > 0 (1.5)_a$$

$$= \frac{\theta^2}{(\theta+1)^{x+2}}\left[1 + \frac{\alpha x + 1}{\theta + \alpha}\right], \ x = 0,1,2,...; \ \theta > 0, \alpha > -\theta (1.6)_a$$

Similarly, the reparamatrization of two parameter Lindley mixture of Poisson distribution becomes

$$f(x; \lambda, \theta) = \int_0^\infty \frac{e^{-\lambda}\lambda^x}{\Gamma(1+x)} \cdot \frac{1}{\theta(1+\alpha\theta)}(1 + \alpha\lambda)e^{-\theta/\lambda}d\lambda, \ x > 0, \alpha > 0, \lambda > 0, \theta > 0 (1.5)_b$$

$$= \frac{\theta^x}{(\theta+1)^{x+2}}\left[1 + \frac{\theta(\alpha x + 1)}{1 + \alpha\theta}\right], \ x = 0,1,2,...; \ \theta > 0, \alpha > -\theta (1.6)_b$$

It can be seen that for α = 1, this distribution reduces to the reparatrization one parameter PLD (1.3). for α = 0, it reduces to the geometric distribution,

$$f\left(x; \lambda, \frac{1}{\theta}\right) = \frac{\theta^x}{(\theta+1)^{x+1}}, \text{ with parameter } p = \frac{\theta}{\theta+1}.$$

MOMENTS

The rth moment about the origin of the reparamatrization two parameter PLD $(1.6)_b$ can be obtained as

$$\mu'_r = E[x^r/\lambda](2.1)$$

From the relation $(1.5)_b$ we get,

$$\mu'_r = \int_0^\infty \left[\sum_{x=0}^\infty x^r \frac{e^{-\lambda}\lambda^x}{\Gamma(1+x)}\right] \cdot \frac{1}{\theta(1+\alpha\theta)}(1 + \alpha\lambda)e^{-\lambda/\theta}d\lambda, \ x > 0, \alpha > 0, \lambda > 0, \theta > 0 (2.2)$$

Obviously the expression under bracket is the rth moment about origin of the Poisson distribution. Taking r = 1, in (2.2) and using the mean of the Poisson distribution, the mean of the reparamatrization discrete two parameter PLD is obtained as

$$\mu'_1 = \frac{1}{\theta(1+\alpha\theta)}\int_0^\infty \lambda(1+\alpha\lambda)e^{-\lambda/\theta}d\lambda$$

$$= \frac{\theta(1+2\alpha\theta)}{\theta(1+\alpha\theta)} \quad (2.3)$$

Taking r = 2,3,4 in (2.2) we get,

$$\mu_2' = \frac{1}{\theta(1+\alpha\theta)} \int_0^\infty (\lambda^2+\lambda)(1+\alpha\lambda)e^{-\lambda/\theta}d\lambda$$

$$= \frac{\theta(1+2\alpha\theta)}{1+\alpha\theta} + \frac{2\theta^2(1+3\alpha\theta)}{1+\alpha\theta}(2.4)$$

$$\mu_3' = \frac{\theta(1+2\alpha\theta)}{\theta(1+\alpha\theta)} + \frac{6\theta^2(1+3\alpha\theta)}{1+\alpha\theta} + \frac{6\theta^3(1+4\alpha\theta)}{1+\alpha\theta}(2.5)$$

and

$$\mu_4' = \frac{\theta(1+2\alpha\theta)}{\theta(1+\alpha\theta)} + \frac{14\theta^2(1+3\alpha\theta)}{1+\alpha\theta} + \frac{36\theta^3(1+4\alpha\theta)}{1+\alpha\theta} + \frac{24\theta^4(1+5\alpha\theta)}{1+\alpha\theta}(2.6)$$

It can be seen that at $\alpha = 1$, the above moments reduce to the respective moments of the reparamatrization one-parameter PLD.

PROBABILITY GENERATING FUNCTION (PGF)

The probability generating function of the discrete two parameter PLD is given by;

$$P_x(t) = E(t^x) = \frac{\theta^2}{(\theta+1)^2} \sum_{x=0} \left(\frac{t}{\theta+1}\right)^x$$

$$+ \frac{\theta^2}{(\theta+1)^2(\theta+\alpha)} \cdot \sum_{x=0} (\alpha x+1)\left(\frac{t}{\theta+1}\right)^x$$

$$= \frac{\theta^2(\theta+1-t)+\alpha\theta^2}{(\theta+1-t)^2(\theta+\alpha)}(2.7)_a$$

Its reparamatrization PGF is given by;

$$P_x(t) = E(t^x) = \frac{\theta^2}{(\theta+1)^2} \sum_{x=0} \left(\frac{t}{\theta+1}\right)^x$$

$$+ \frac{\theta^2}{(\theta+1)^2(\theta+\alpha)} \cdot \sum_{x=0} (\alpha x+1)\left(\frac{t}{\theta+1}\right)^x$$

$$= \frac{1+\theta-t\theta+\alpha\theta}{(\theta+1-t\theta)^2(1+\alpha\theta)}(2.7)_b$$

MOMENT GENERATING FUNCTION (MGF)

The moment generating function of the discrete two parameter PLD is given by

$M_x(t) = E(e^{tx}) = \frac{\theta^2(\theta+1-e^t)+\alpha\theta^2}{(\theta+1-e^t)^2(\theta+\alpha)}(2.8)_a$ and its reparamatrization MGF is given by;

$$M_x(t) = \frac{1+\theta-\theta e^t+\alpha\theta}{(\theta+1-\theta e^t)^2(1+\alpha\theta)}(2.8)_b$$

ESTIMATION OF PARAMETER

In this section we derive estimators for the two parameter α and $1/\theta$ we use two methods

ESTIMATION BASED ON THE METHOD OF MOMENTS:

By using the relation (2.3) and (2.4) we get;

$$\frac{\mu_2'-\mu_1'}{\mu_1'^2} = \frac{2(1+3\alpha\theta)(1+\alpha\theta)}{(1+2\alpha\theta)^2} = k \text{ (say)} \qquad (3.1)$$

Setting,

$$\frac{1}{\theta} = b\alpha \quad \text{or } \frac{1}{b} = \alpha\theta \text{ in (3.1) we get;}$$

$$\frac{\mu_2'-\mu_1'}{\mu_1'^2} = \frac{2(b+3)(b+1)}{(b+2)^2} = m \text{ (say)} \qquad (3.2)$$

Or, $2b^2+8b+6 = mb^2+4mb+4m$

Or, $(2-m) b^2 + (8-4m) b + (6-4m) = 0 \qquad (3.2)$

Which is a quadratic equation in b.

Replacing the first two population moments by the respective sample moments in (3.1) an estimate k of m can be obtained. Using m in (3.2), an estimate $\hat{b}$ of b, can be obtained. It can be seen that estimates of b can be obtained from (4.2) only when m<2.

Again, substituting $\frac{1}{\theta} = b\alpha$ or, $\frac{1}{b} = \alpha\theta$ in the expression for mean (2.3) we get,

$$\frac{\frac{1}{b\alpha}(1+2b)}{1+b} = \frac{(1+2b)}{\alpha b(b+1)} = \bar{X}, \text{ and thus an estimator of } \alpha \text{ and } \theta \text{ are obtained as:}$$

$$\hat{\alpha} = \left(\frac{1+2b}{b(1+b)}\right)\frac{1}{\bar{X}} \hat{\beta} = \frac{1}{b\hat{\alpha}} = \left(\frac{b+1}{1+2b}\right)\bar{X}. \qquad (3.3)$$

ESTIMATION USING THE MAXIMUM LIKELIHOOD METHOD:

Let, $x_1, x_2, \ldots, x_n$ be a random sample of size n from the reparamatrization of the discrete two-parameters PLD $(1.6)_b$ and let f_x be the observed frequency in the sample corresponding to $X = x$ (x=1,2,…,k) such that $\sum_1^k f_x = n$, where k is the largest observed value having non-zero frequency. The likelihood function, L of the reparamatrization of the discrete two-parameter PLD $(1.6)_b$ is given by

$$L = \frac{\theta^{\sum_{x=1}^k x}}{(1+\theta)^{\sum_{x=1}^k (x+2)f_x}} \prod_{x=1}^k [(\theta+1) + \alpha\theta(x+1)]^{f_x} \qquad (3.4)$$

and the log likelihood function becomes

$$\log L = n\bar{x}log\theta - \sum_{x=1}^{k} f_x (x+2)\log(\theta+1) + \sum_{x=1}^{k} f_x \log[(\theta + 1) + \alpha\theta(x+1)] \tag{3.5}$$

The resultant two log likelihood equations are thus obtained as:

$$\frac{\partial logL}{\partial \theta} = \frac{n\bar{x}}{\theta} + \sum_{x=1}^{k} \frac{\alpha(x+1)f_x}{[(\theta+1)+\alpha\theta(x+1)]} = 0 \tag{3.6}$$

and,

$$\frac{\partial logL}{\partial \alpha} = \sum_{x=1}^{k} \frac{\theta(x+1)f_x}{[(\theta+1)+\alpha\theta(x+1)]} = 0 \tag{3.7}$$

The two equations (3.6) and (3.7) given above do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. For this, we have:

$$\frac{\partial^2 logL}{\partial \theta^2} = -\frac{n\bar{x}}{\theta^2} - \sum_{x=1}^{k} \frac{\alpha(x+1)\{1+\alpha(x+1)\}f_x}{[(\theta+1)+\alpha\theta(x+1)]^2} \tag{3.8}$$

$$\frac{\partial^2 logL}{\partial \theta \partial \alpha} = \sum_{x=1}^{k} \frac{(x+1)f_x}{[(\theta+1)+\alpha\theta(x+1)]^2} \tag{3.9}$$

$$\frac{\partial^2 logL}{\partial \alpha^2} = -\sum_{x=1}^{k} \frac{\{\theta(x+1)\}f_x}{[(\theta+1)+\alpha\theta(x+1)]^2} \tag{3.10}$$

The equations can be solved by an iteration procedure to obtain maximum likely hood estimators of $1/\hat{\theta}$ and $\hat{\alpha}$ starting with initial values θ_0 and α_0 of θ and α, respectively.

$$\begin{bmatrix} \dfrac{\partial^2 logL}{\partial \theta^2} & \dfrac{\partial^2 logL}{\partial \alpha \partial \theta} \\ \dfrac{\partial^2 logL}{\partial \alpha \partial \theta} & \dfrac{\partial^2 logL}{\partial \alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial logL}{\partial \theta} \\ \dfrac{\partial logL}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

CONCLUSION

In this paper, the writer proposed a reparamatrization of a two-parameter PLD of which one parameter is a particular case, for modeling waiting and survival times data. Several properties of this PLD such as moments, probability generating function, moment generating function and estimation of parameters by using method of moments and the method of maximum likelihood have been discussed. This distribution can be fitted by using chi-square test for appropriate waiting and survival times data.

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