ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS WITH FOURTH-ORDER CONVERGENCE

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ABSTRACT

In this paper, we obtain fourth order iterative method for solving nonlinear equations by combining arithmetic mean Newton method, harmonic mean Newton method and midpoint Newton method uniquely. Also, some variant of Newton type methods based on inverse function have been developed. These methods are free from second order derivatives.

Key Words: Newton method, nonlinear equation, fourth-order convergence, inverse function, iterative method.

INTRODUCTION

Nonlinear equations play important role in many branches of science and engineering. Finding an analytic solution to nonlinear equations is not always possible. So finding numerical solution of nonlinear equations become important research in numerical analysis. In this paper, we consider the iterative methods to find the simple root of nonlinear equations

$$f(x) = 0 \tag{1}$$

where $f : D \subset R \rightarrow R$ for an open interval D is a scalar function.

One of the most widely used numerical method is Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

This is an important and basic method (Bradie, 2007) which converges quadratically. In the recent years, a tremendous variant of this method has appeared showing one or the other advantages over this method in some sense.

DEFINITION: (Weerakoon and Fernando, 2002).If the sequence $\{x_n | \, n \geq 0\}$ tends to a limit α in such a way that

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$$\lim_{x_n \to \alpha} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = c\,,$$

for some $c \neq 0$ and $p \geq 1$, then the order of convergence of the sequence is said to be p and c is known as asymptotic error constant.

When $p = 1$, the convergence is linear, and it is called the first order convergence. If $p = 2$ and $p = 3$, the sequence is said to converge quadratically and cubically respectively. The value of p is called the order of convergence of the method which produce the sequence $\{x_n |\, n \geq 0\}$. Let $e_n = x_n - \alpha$ is the error in nth iterate. Then the relation

$$e_{n+1} = ce_n^p + O(e_n^{p+1})$$

is called the error equation for the method, p being the order of convergence.

DEFINITION: (Singh, 2009). Efficiency index is simply define as $p^{\frac{1}{m}}$, where p is the order of convergence of the method and m is the number of the function evaluations required by the method per iteration. The efficiency index of Newton method is 1.41 and secant method is 1.62.

SOME VARIANT OF NEWTON METHOD

Weeraken and Fernando (2000) used the Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt \tag{3}$$

and approximate the integral by trapezoidal rule that is

$$\int_{x_n}^{x} f'(t)dt = \frac{(x - x_n)}{2}[f'(x_n) + f'(x)]. \tag{4}$$

Then we obtained the variant of Newton method which is given by the formula

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(x_n) + f'(x_n^*)]}, \tag{5}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

The method (5) can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{[f'(x_n) + f'(x_n^*)]}{2}}$$

This method is called arithmetic mean Newton method since this variant of Newton method can be viewed as obtained by using arithmetic mean of $f'(x_n)$ and $f'(x_n^*)$ instead of $f'(x_n)$ in Newton method (2). If we approximate the indefinite integral in equation (3) by using the harmonic mean ($\ddot{O}$zban, 2004; Ababneh, 2012) that is if we use the harmonic mean instead of the arithmetic mean in equation (3), we get

$$x_{n+1} = x_n - \frac{f(x_n)[f'(x_n)+f'(x_n^*)]}{2f'(x_n)f'(x_n^*)} \tag{6}$$

Also if we approximate the indefinite integral in equation (3) by midpoint rule (Ababneh, 2012; Jain, 2013)

$$\int_{x_n}^{x} f'(t)dt = (x - x_n)f'\left(\frac{x + x_n}{2}\right)$$

We obtain the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}, \tag{7}$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

This method is called midpoint Newton (MN) method.

COMBINATION OF METHODS

Multiplying equation (5), (6), (7) by $a,\ b, c$ respectively and adding, where $a + b + c = 1$, we get

$$x_{n+1} = x_n - \left[a\frac{2f(x_n)}{f'(x_n^*) + f'(x_n^*)} + b\frac{f(x_n)[f'(x_n) + f'(x_n^*)]}{2f'(x_n)f'(x_n^*)} + c\frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}\right]$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$. (8)

We shall prove here that all the methods given by families of method (8) are of order at least 3 and for unique values of a, b, c, the resulting method is of order 4. We begin with the following:

THEOREM: Let the function f has sufficient number of continuous derivatives in a neighborhood α which is a simple zero of f, that is, $f(\alpha)=0, f'(\alpha)\neq 0$. Then, all the methods given by the family of method (8) are of order 3 and for unique value of $a=\frac{-2}{3}$, $b=1, c=\frac{2}{3}$, we get the method is of order 4.

Proof. Let α be a simple root of f (x) = 0 (i.e.$f(\alpha)=0$ and $f'(\alpha)\neq 0$) and e_n be the error in nth iterate. Then using the Taylor's expansions and after some calculation ($\ddot{O}$zban, 2004; Weerakoon and Fernando, 2012), we get

$$\frac{2f(x_n)}{[f'(x_n)+f'(x_n^*)]}=e_n-\left(c_2^2+\frac{1}{2}c_3\right)e_n^3+O(e_n^4) \tag{9}$$

$$\frac{f(x_n)[f'(x_n)+f'(x_n^*)]}{2f'(x_n)f'(x_n^*)}=e_n-\frac{1}{2}c_3e_n^3+O(e_n^4), \tag{10}$$

and

$$\frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}=e_n+(\frac{1}{4}c_3-c_2^2)\,e_n^3+O(e_n^4) \tag{11}$$

where $c_j=\frac{1}{j!}\frac{f^j(\alpha)}{f'(\alpha)}$, $j=2,3,\ldots.$

Substituting the values from (9), (10) and (11) in (8), we get

$$\begin{aligned} e_{n+1} &= e_n-(a+b+c)e_n+a\left(c_2^2+\frac{1}{2}c_3\right)e_n^3+b\left(\frac{1}{2}c_3e_n^3\right)-c \\ &\quad +(\frac{1}{4}c_3-c_2^2)\,e_n^3+O(e_n^4) \\ &= \left(\frac{a}{2}+\frac{b}{2}-\frac{c}{4}\right)c_3e_n^3 \;+(\mathrm{a}+\mathrm{c})c_2^2e_n^3+O(e_n^4) \end{aligned}$$

Hence from above, rate of convergence of each method given by the family of method (8) is at least three and we get the method is of order four for unique value of $a=\frac{-2}{3}$, $b=1, c=\frac{2}{3}$. Thus the method for order convergence four is

$$x_{n+1}=x_n+\frac{4}{3}\frac{f(x_n)}{f'(x_n)+f'(x_n^*)}-\frac{f(x_n)[f'(x_n)+f'(x_n^*)]}{2f'(x_n)f'(x_n^*)}-\frac{2}{3}\frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)},$$

where $x_n^*=x_n-\frac{f(x_n)}{f'(x_n)}$. (12)

This method (12) is same as Dehghan and Hajarian method (2010) but they approximated the indefinite integral in Newton's theorem by linear combination of trapezoidal integration rule, midpoint integral rule and harmonic mean rule and there is no idea how they choose constants.

METHODS BASED ON INVERSE FUNCTION

In this section, we use (Jain, 2013) $x = f^{-1}(y)$ = g(y) instead of $y = f(x)$, we obtain

$$g(y) = g(y_n) + \int_{y_n}^{y} g'(t)dt \qquad (13)$$

If we approximate the indefinite integral in (13) by harmonic mean rule, we get

$$\int_{y_n}^{y} g'(t)dt = (y - y_n)\, \frac{2g'(y_n)g'(y)}{[\, g'(y_n)+g'(y)]}. \qquad (14)$$

Hence from (13)

$$g(y) = g(y_n) + (y - y_n)\, \frac{2g'(y_n)g'(y)}{[\, g'(y_n)+g'(y)]}$$

where $y_n = f(x_n)$. Now using the fact that $g'(y) = (f^{-1})'(y) = [f'(x)]^{-1}$ and that

$y = f(x) = 0$, we obtain

$$x = x_n + \big(0 - f(x_n)\big)2\, \frac{\frac{1}{f'(x_n)}\frac{1}{f'(x)}}{\frac{1}{f'(x_n)} + \frac{1}{f'(x)}}$$

$$= x_n - \frac{2f(x_n)}{[f'(x_n) + f'(x)]}$$

Thus when $x \rightarrow x_{n+1}$ and in right side if we use $x_n^* = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, then we get the iterative formula

$$x_{n+1} = x_n - \frac{2f(x_n)}{[f'(x_n)+f'(x_n^*)]}\,. \qquad (15)$$

This formula is exactly same as the formula (5) obtained by approximating the indefinite integral of equation (3) using the trapezoidal rule for the function

$y = f(x)$.

Again if we approximate the indefinite integral in equation (13) by midpoint rule, we get

$$\int_{y_n}^{y} g'(t)dt = (y - y_n)g'\left(\frac{y + y_n}{2}\right)$$

Hence from equation (13),

$$g(y) = g(y_n) + (y - y_n)g'\left(\frac{y + y_n}{2}\right)$$

$$x = x_n + \left(0 - f(x_n)\right)\frac{1}{f'\left(\frac{x+x_n}{2}\right)}$$

Therefore iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\frac{x_n+x_n^*}{2}\right)}, \qquad (16)$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$. This method is exactly same as method given by equation (7).

Finally if we approximate the indefinite integral in equation (13) by trapezoidal rule, we get

$$\int_{y_n}^{y} g'(t)dt = \frac{(y - y_n)}{2}[g'(y_n) + g'(y)]$$

Also from (13), we get

$$g(y) = g(y_n) + \frac{(y - y_n)}{2}[g'(y_n) + g'(y)]$$

or $$x = x_n + \frac{(0 - f(x_n)}{2}\left[\frac{1}{f'(x_n)} + \frac{1}{f'(x)}\right]$$

$$= x_n - \frac{f(x_n)[f'(x_n) + f'(x_n^*)]}{2f'(x_n)f'(x_n^*)}$$

Therefore iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)[f'(x_n) + f'(x_n^*)]}{2f'(x_n)f'(x_n^*)}, \qquad (17)$$

where $x_n^* = x_n - \frac{f(x_n)}{f'(x_n)}$.

This formula is same as the formula (6) obtained by approximating the indefinite integral of equation (3) using the harmonic mean rule for the function $y = f(x)$. From above it is clear that the fourth-order convergence method based on inverse function obtained by combining the methods which are obtained respectively by approximating the indefinite integral of Newton's formula by harmonic mean rule, midpoint rule and trapezoidal rule is also given the same formula as (12).

CONCLUSION

From above discussion, it is clear that order of convergence of the numerical method for finding the simple root of nonlinear equation $f(x) = 0$ obtained from the combination of arithmetic mean Newton method, harmonic mean Newton method and midpoint Newton method is at least three and it become four for unique combination. Also we conclude that numerical methods obtained by using inverse function $x = f^{-1}(y) = g(y)$ instead of $y = f(x)$ and approximating the indefinite integral in Newton's theorem by trapezoidal integration rule, harmonic mean rule, midpoint rule are same as harmonic mean Newton method, arithmetic mean Newton method and midpoint Newton method respectively. Thus, the method obtained by the combination of methods obtained by using inverse function $x = f^{-1}(y) = g(y)$ instead of $y = f(x)$ and approximating the indefinite integral in Newton's theorem by trapezoidal integration rule, harmonic mean rule and midpoint rule is same as method (12). This method is also free from second order derivatives.

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