Entropy of coupled Klein-Gordan field in FLRW space-time

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Abstract: The coupled Klein-Gordan (KG) field in FLRW space-time has been further studied to explore its effect on entropy. The modifications in pressure and density and hence entropy due to coupling have been calculated. There is an indication seen that the introduction of the coupling contributes to the generation of entropy in the KG field which may probably explain the large entropy of the universe.

Keywords: FLRW space-time; Klein-Gordan field; Entropy; NP formalism.

Introduction

In the course of studying density perturbation in the context of studying and explaining the formation and evolution of the large scale structures in the Universe[1-6], investigating the behaviour of the massive Klein-Gordan (KG) field and the Dirac field coupled to the Friedman Lemaitre Robertson Walker (FLRW) spacetime has been useful. In our previous works[7-11], we have attempted to do so using Neumann-Penrose (NP) formalism[12,13], where the equation was solved separating the angular, radial and temporal parts. WKB approximation up to the first order was used for the temporal part to explain the evolution of the universe. In this work, we are further investigating the coupled KG field to see its effect on entropy.

Separation of the KG field equations

Since it is the extension of our previous works[7], some equations have been gathered here from the work for ready reference. The metric for FLRW space-time can be written as

$$ds^2 = a^2[d\eta^2 - dr^2 - S^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \quad \dots (1)$$

where a is the scale factor which is connected to the conformal time η and the co-moving time t by $dt = a\, d\eta$, and

$$S = \frac{\sin\sqrt{k}r}{\sqrt{k}} = \begin{cases} \sin r, & k = 1 \text{ for closed} \\ r, & k = 0 \text{ for flat} \\ \sinh r, & k = -1 \text{ for open} \end{cases} \quad \dots (2)$$

From the equation of motion

$$[\Box^2 + M^2 + \xi R]\psi = 0 \quad \dots (3)$$

where ψ can be separated into its spatial and temporal parts as $\psi(\eta, r) = \mathbb{T}(\eta)f(r)$ and hence we can write

$$\left[\frac{1}{a^2}\partial_\eta a^2 \partial_\eta + M^2a^2 + \xi R a^2\right]\psi = \nabla^2\psi = \text{Constant}\,\psi = (k - p^2)\psi \quad \dots (4)$$

Here the separation constant p^2 can identified with the co-moving momentum: $k = pa$ and ξ as the coupling constant.

This can further be written as

$$\left[\partial_\eta{}^2 + M^2a^2 + (\xi - \tfrac{1}{6})Ra^2\right]aS\psi = \left[\partial_r{}^2 + \frac{1}{S^2}\nabla^2_{\theta\phi}\right] aS\psi = (\text{Constant} - k)\,\psi = (-p^2)\psi \quad \dots (5)$$

Rearranging and using the expressions of energy momentum $T^\mu{}_\mu$ and its time and spatial components $a^4T^\eta{}_\eta$ and $-aT^i{}_i$, we get

$$D_\mu T^\mu{}_\nu = -2\xi R^\mu{}_\nu D_\mu|\psi|^2 \quad \dots (6)$$

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The separated radial and angular parts of ψ will be removed using the normalization condition:

$\int |f(r)|^2 \sin^2 r \sin\theta \, dr \, d\theta \, d\phi = 1$, and leaving only the temporal part $\mathbb{T}$:

$$\frac{1}{a^4}\frac{\partial}{\partial\eta} a^4 \, T^{\eta}{}_{\eta} + \frac{a\prime}{a} T^i_i = -2\xi R^{\eta}{}_{\eta} \frac{\partial}{\partial\eta} |\mathbb{T}|^2 \qquad \dots (7)$$

Converting $d\eta$ to da (since $\frac{\partial a}{\partial\eta} = a'$), replacing T^i_i by ρ - 3P and rearranging the right hand side,

$$\frac{1}{a^3}\frac{\partial}{\partial a} a^3 \, T^{\eta}{}_{\eta} + \frac{3P}{a} = -2\xi \frac{\partial}{\partial a}\left(a^3 R^{\eta}{}_{\eta} |\mathbb{T}|^2\right) + 2\xi \, |T|^2 \frac{\partial}{\partial a}\left(a^3 \, |\mathbb{T}|^2\right) \qquad \dots (8)$$

The various values to be used are listed below

$$a'^2 = \left(\frac{da}{d\eta}\right)^2 = H_0^2[\Omega_\Lambda a^4 + \Omega_K a^2 + \Omega_R] \; \dots (9)$$

which is given by the Friedman-Lemaitre equation such that Ω_Λ, Ω_k, and Ω_R are densities contributed by the cosmological constant, curvature and radiation, respectively, in units of the critical density at the time η_0 when the scale factor is normalized to $a_0 = 1$;

$$a'' = H_0^2[2\Omega_\Lambda a^3 + \Omega_K a] \qquad \dots (10)$$

$$a''' = H_0^2[6\Omega_\Lambda a^2 + \Omega_K] \, a' \qquad \dots (11)$$

$$a^2 R^{\eta}{}_{\eta} = R_{\eta\eta} = 3\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right) \qquad \dots (12)$$

$$= 3H_0^2\left[\Omega_\Lambda a^2 - \frac{\Omega_R}{a^2}\right] \text{ (using Eq. (8) \& (9))}$$

$$a^2 T^{\eta}{}_{\eta} = T_{\eta\eta} = a^2\rho = na^2\varepsilon \qquad \dots (13)$$

Normalizing the total comoving particle number as

$Na^3 = 2\pi^2 na^3 = 1$, the comoving particle density is obtained as

$na^3 = 1/2\pi^2$ and the comoving energy density as

$$\rho a^3 = na^3\varepsilon = \left(\frac{1}{2\pi^2}\right)\varepsilon \qquad \dots (14)$$

The temporal wave function is given by

$$|a\mathbb{T}|^2 = \frac{Na^3}{2a\varepsilon} \Rightarrow |\mathbb{T}|^2 = \frac{1}{2a^2(a\varepsilon)} \qquad \dots (15)$$

The comoving total energy $a\varepsilon$ is given by

$$(a\varepsilon)^2 = p^2 + a^2M^2 + 3\xi\left(\frac{a''}{a} + \frac{a'^2}{a^2} + 2K\right) \; \dots (16)$$

Using Eq. (8) and (9), this becomes

$$(a\varepsilon)^2 = p^2 + a^2M^2 + 3\xi H_0^2\left[\Omega_\Lambda a^2 + \frac{\Omega_R}{a^2}\right] \; \dots (17)$$

The pressure P is given by

$$\rho - 3P = \frac{nM^2}{\varepsilon} \Rightarrow 3P = \rho - \frac{M^2}{2\pi^2 a^2(a\varepsilon)} \; \dots. (18)$$

Entropy of coupled kg field

We now use the information/relations gathered in the previous section to calculate the entropy of coupled KG fields / particles. Using Eq. (12), (9) and (10) in (8), we get

$$d\left[a^3\left\{T^{\eta}{}_{\eta} + 6\xi \, H_0^2 \left(\Omega_\Lambda a^3 - \frac{\Omega_R}{a}\right)|\mathbb{T}|^2\right\}\right] + da^3\left[P + 2\xi \, H_0^2 \, |T|^2\left(3\Omega_\Lambda a^3 + \frac{\Omega_R}{a}\right)\right] \qquad \dots (19)$$

which is in the form $d(a^3\rho_{\text{eff}}) + P_{\text{eff}} \, da^3 = 0 \dots (20)$

where,

$$\rho_{\text{eff}} = T^{\eta}{}_{\eta} + 6\xi \, H_0^2 \left(\Omega_\Lambda a^3 - \frac{\Omega_R}{a}\right)|\mathbb{T}|^2 \; \dots (21)$$

$$P_{\text{eff}} = P + 2\xi \, H_0^2 \, |\mathbb{T}|^2\left(3\Omega_\Lambda a^3 + \frac{\Omega_R}{a}\right) \; \dots (22)$$

Some typical plots of them have been shown in Fig.-1 and Fig.-2.

The Eq. (19) is the standard entropy conservation equation: $d(\rho \, V) + P \, dV = \mathcal{T} \, d\mathcal{S} = 0$ so that the entropy is given by

$$\mathcal{S} = \frac{V\,(\rho + P)}{\mathcal{T}} \qquad \dots (23)$$

Since the temperature, $\mathcal{T} \sim 1/a$, for the coupled KG fields/particles, the effective entropy per comoving volume is given by $\mathcal{S}_{\text{eff}} = a^4(\rho_{\text{eff}} + P_{\text{eff}}) \quad \dots (24)$

Its typical plots have been shown in Fig.-3. Interestingly, the curve is flat across a short period around a = 0.5 (where we have taken a = 1 at around equilibrium between cosmological constant and radiation with small contribution of curvature) and then rises up. Also the entropy is overall larger for larger coupling constant.

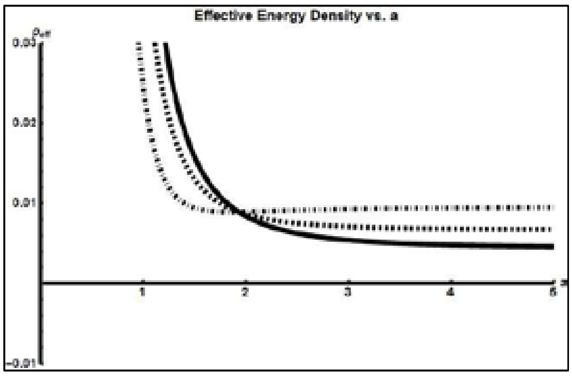

Figure 1: Effective energy density of coupled KG fields. The solid curve is for the coupling constant, $\xi < 1/6$, dotted one is for $\xi = 1/6$ and dot-dashed for $\xi > 1/6$. In the early period, the energy density with lower coupling constant has dominated whereas for later epochs, larger coupling constant has contributed to larger energy density.

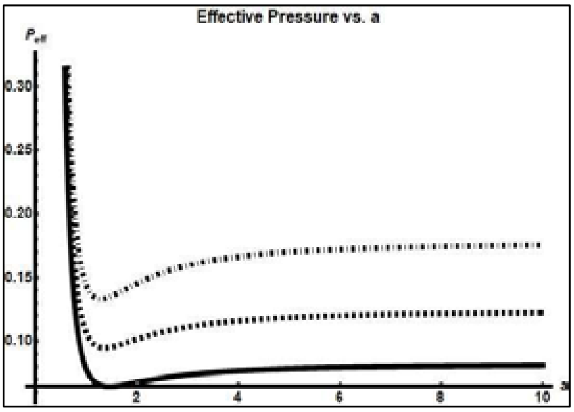

Figure 2: Effective pressure variation with scale factor at various coupling constant. The solid curve is for the coupling constant, ξ < 1/6, dotted one is for ξ = 1/6 and dot-dashed for ξ > 1/6. Larger the coupling constant, larger the pressure.

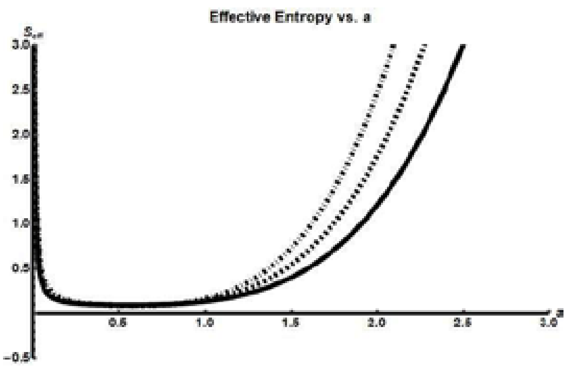

Figure 3: Effective entropy of coupled KG field per comoving volume at various coupling constant. The solid curve is for the coupling constant, ξ < 1/6, dotted one is for ξ = 1/6 and dot-dashed for ξ > 1/6. The effective entropy is seen to grow towards both small and larger a while remaining constant at small middle region abound *a* = 0.5 where we have taken *a* = 1 at around $\Omega_\Lambda \approx \Omega_R \approx 50\%$.

Conclusions

In this work, we have attempted to interpret the conservation of the energy-momentum of massive Klein-Gordon field coupled to a background FLRW spacetime. The coupling is seen to introduce viscous like terms consisting of one part that can be added to the comoving density as an energy generated by the viscous flow, and the remaining part can be added to pressure which can be considered as an additional pressure offered to the viscous fluid. Then we see that the comoving entropy can be written as $\mathcal{S}_{\mathrm{eff}} = a^4(\rho_{\mathrm{eff}} + P_{\mathrm{eff}})$; where it should be noted that this is basically the familiar form with temperature T ~ $1/a$: The plot for S reveals that it grows at both small and large *a*; and becomes flat at mid region. The plot of the pressure has also shown interesting trend. While it is high at small *a* and almost flat for large *a*, it goes to a minimum at some value at the middle. Further detailed investigation must be done to find out whether this process can account for the large entropy of the Universe[14].

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