

A brief review on the solutions of advection-diffusion equation

P. N. Gautam^{*}, B. P. Sapkota^{**} and K. N. Uprety^{***}

^{*} Patan Multiple Campus, Tribhuvan University, Kathmandu, Nepal.

^{**} Ratna Rajyalaxmi Campus, Tribhuvan University, Kathmandu, Nepal.

^{***} Central Department of Mathematics, Tribhuvan University, Kathmandu, Nepal.

Abstract: In this work both linear and nonlinear advection-diffusion equations are considered and discussed their analytical solutions with different initial and boundary conditions. The work of Ogata and Banks, Harleman and Rumer, Cleary and Adrian, Atul Kumar et al., Mojtabi and Deville are reviewed for linear advection-diffusion equations and for nonlinear, we have chosen the work of Sakai and Kimura. Some enthusiastic functions used in the articles, drawbacks and applications of the results are discussed. Reduction of the advection-diffusion equations into diffusion equations make the governing equation solvable by using integral transform method for analytical solution. For nonlinear advection-diffusion equations, the Cole-Hopf transformation is used to reduce into the diffusion equation. Different dispersion phenomena in atmosphere, surface and subsurface area are outlined.

Keywords: Cole-Hopf transformation; Integral transform; Dispersion; Viscosity.

Introduction

Adolf Fick described the law of diffusion and derived the

differential equation $\frac{\partial y}{\partial t} = k \frac{\partial^2 y}{\partial x^2}$, where k is diffusivity,

$y = f(x, t)$ is a function of two independent variables x and t . Advection diffusion equation (ADE) was developed later, by adding the advection term on it. ADE describes the transport of solute particles, energy or other physical quantities due to flow of fluid which induces fluxes of energy and matter. Advection is due to ambient flow of matter or energy and diffusion is due to random motion of the particles which occurs even if the fluid is at rest and net transport takes place under certain conditions. ADE represents the condition when both diffusion and advection (or convection) take place simultaneously⁸. Mixing of fluid and pollutant in the river, adding more smokes in air, issues

of climate change and use of advanced computers for simulation of real world problems increases the importance of ADE in recent and active research area. One dimensional linear ADE is comparatively easy to solve and realistic result is obtained in most of the cases. In the study of the transport of dissolve particles with fluctuated velocity especially when the flow rate is very low and for the area which is more polluted, ADE plays an important role. ADE also helps to describe ecosystem in certain area and in the modeling of many biological processes. ADE is used to study the transient mass transfer phenomena occurring in natural as well as artificial systems and helps to study the problems related to the movement of diffusible particles, ions, chemicals etc. So, ADE is an important tool to understand, solve and describe the real world problems with physical phenomena⁹.

Author for correspondence: P. N. Gautam, Patan Multiple Campus, Tribhuvan University, Kathmandu, Nepal.
Email: gtmpushpal@gmail.com

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Ogata and Banks⁶ and Harleman and Rumer³ have used ADE to study the dispersion in porous media. Cleary and Adrian² have applied ADE to study dispersion of cation in soil column. Atul Kumar et al.⁴ have used ADE to study river pollutions and transport of chemicals in subsurface areas with low diffusivity. Atul Kumar et al. have discussed the relationship between diffusivity and seepage velocity. Abdelkadir and Michel⁵ has decomposed the equation into advection and diffusion and solved them separately. Harleman and Rumer³, have used a different idea to calculate longitudinal and lateral dispersion separately. Sakai and Kimura have taken nonlinear advection equation and reduced it into linear diffusion equation by using Cole-Hopf transformation. Distortion occurs in the signal while propagating in case of nonlinear ADE. Shocking and rarefaction effect is the work of uu_x which causes distortion.

Governing equation and solutions

Dispersion in porous medium

Ogata and Bank⁶ have taken the ADE as a governing

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = D \frac{\partial^2 c}{\partial x^2} \quad \dots (1)$$

equation where, C is concentration, u constant advective velocity, D is diffusion or dispersion coefficient and initial concentration is considered as C_0 . By introducing new variable, the ADE is reduced into the diffusion equation and Duhamel's theorem is used to get analytical solution which is

$$C(x, t) = C_0 \frac{2}{\sqrt{\pi}} \exp\left(\frac{u^2 t}{4D}\right) \times \left\{ \int_0^\infty \exp(-\lambda^2 - \epsilon^2 / \lambda^2) d\lambda - \int_0^\infty \exp(-\lambda^2 - \epsilon^2 / \lambda^2) d\lambda \right\} \quad \dots (2)$$

Where, $\epsilon = \frac{ux}{4D}$ and $\alpha = \frac{x}{2\sqrt{Dt}}$ ⁶. This solution is further simplified and expressed in the form of error function which is $\frac{c}{c_0} = \frac{1}{2} \left[\operatorname{erfc}\left(\frac{x-ut}{2\sqrt{Dt}}\right) + \exp\left(\frac{ux}{D}\right) \operatorname{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right) \right]$

.... (3)

If $\xi = \frac{ut}{x}$ and $\eta = \frac{D}{ux}$, two dimensionless quantities are introduced the solution reduces to

$$\frac{c}{c_0} = \frac{1}{2} \left[\operatorname{erfc}\left(\frac{1-\xi}{2\sqrt{\xi\eta}}\right) + \exp\left(\frac{1}{\eta}\right) \operatorname{erfc}\left(\frac{1+\xi}{2\sqrt{\xi\eta}}\right) \right] \quad \dots (4)$$

It is claimed that the solution does not work for all conditions⁶. If the boundaries are symmetrical or η is nearly zero in the Figure 1, only first term of equation (4) gives the correct result. The second term is used for general type of problems having asymmetrical boundary conditions or η becomes large as shown in Figure 1.

When the second term is included, error up to five percent occurs for general type of problems. So the major focus is applied to find out the influence of the second term. Taking it as a single function and using the method of maxima and minima value of ξ is calculated. The conclusion is that the second term becomes most significant when $\xi = 1$. But when we use $\xi = 1$ in equation (4) we get:

$$\frac{c}{c_0} = \frac{1}{2} [1 + \text{second term}] \quad \dots (5)$$

which clearly tells that the second term is most significant when $\xi = 1$ which can be stated without doing the process of maxima and minima. It is not possible to repeat the experiment mentioned in the article but the authors have reported that when diffusivity ranges from 10^{-4} cm²/sec to 10^{-2} cm²/sec then the diffusivity is proportional to velocity. It is also mentioned that for the values of $D/ux = \eta < 0.002$ a maximum error of less than 3 percent occurs when the second term is neglected. Figure 1 shows that the condition approaches towards symmetry as $\eta \rightarrow 0^6$. Harleman and Rumer³ have taken two dimensional ADE with unidirectional flow which is

$$\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} = D_1 \frac{\partial^2 S}{\partial x^2} + D_2 \frac{\partial^2 S}{\partial y^2} \quad \dots (6)$$

Where, D_1 and D_2 are longitudinal and lateral dispersion respectively with advective velocity u . It is assumed that the flow is steady and no dispersion in y direction to calculate D_1 . The solution obtained is same as (4). For the case of lateral dispersion it is assumed that $\frac{\partial^2 S}{\partial x^2} \ll \frac{\partial^2 S}{\partial y^2}$ then the equation is reduces to $u \frac{\partial S}{\partial x} = D_2 \frac{\partial^2 S}{\partial y^2}$

.... (7)

This equation (7) is solved by using Fourier transform method and the solution is verified by using experimental method.

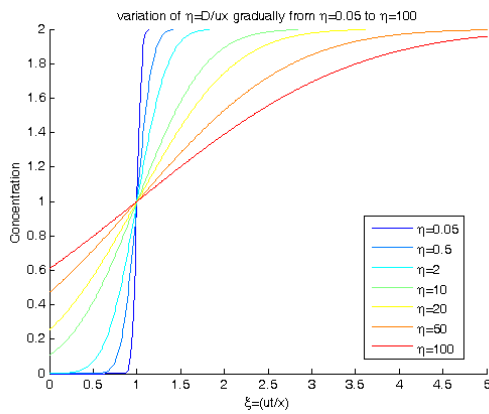


Figure 1: Concentration distribution as per variation of η from 0.05 to 100.

In this article Harleman and Rumer³ have explained a very sophisticated experiment performed at the Hydrodynamics laboratory of Massachusetts Institute of Technology in which particles of different shape and size with varying pore system is considered. We have no alternative except to accept the result as reported. This article also mentioned the result of Scheidegger³ which tells that the dispersion is related to pore system geometry and seepage velocity which can be written as:

$$D_1 = \alpha_1 |u| \text{ and } D_2 = \alpha_2 |u|$$

where, α_1 and α_2 are constant of proportionality and $|u|$ is seepage velocity³. Depending upon the pore-system geometry, α_1 is proportional to the average grain size of the porous medium and the relation established by Raimondi et al. and Mehlhorn³ states that $\alpha_1 = \beta d$, where d is average grain size and β is constant of proportionality.

Substituting this value of α_1 and dividing both side by kinematic viscosity ν

$$\frac{D_1}{\nu} = \left(\frac{\beta d}{\nu}\right) |u| = \left(\frac{d|u|}{\nu}\right) \beta = \beta R \quad \dots (8)$$

where, $R = \frac{d|u|}{\nu}$ is Reynolds number. There is no further discussion about this result but another similar result of Harleman, Mehlhorn and Rumer³ is taken, which is $\frac{D_1}{\nu} = \psi_1 R^{n_1}$ and $\frac{D_2}{\nu} = \psi_2 R^{n_2}$ (9)

Here ψ_1 is as β in the equation (8) and both ψ_1 and n_1 are independent of particle size for nearly uniform media.

$$\frac{D_1}{D_2} = \frac{\psi_1}{\psi_2} R^{n_1 - n_2} \quad \dots (10)$$

From experiment it is established that the value of n_1 is 1.2 and may approach 1 for highly non-uniform media. So we can say that $1 < n_1 < 1.2$. But nothing is mentioned about n_2 which is related to lateral dispersion. It is not explained how n_2 differs with n_1 and if they are identical we cannot calculate the Reynolds number. Major findings of this article are

$$\frac{D_1}{\nu} = 0.66R^{1.2} \quad \text{and} \quad \frac{D_2}{\nu} = 0.36R^{0.7} \quad \text{which implies } \frac{D_1}{D_2} =$$

$$18.33R^{0.5} . \text{ For a porous medium when } R = 0.1, \frac{D_1}{D_2} = 5.8$$

$$\text{and when } R = 1, \frac{D_1}{D_2} = 18.3 \text{ which shows that the ratio}$$

increases with increase in R . From the relation $R = u d/\nu$ where both, average grain size (d) and viscosity (ν) are constant so $R \propto u$. This shows that the ratio increases with increase in seepage velocity but not justified for Reynolds number greater than 10. According to Bischoff and Levenspiel, if the flow is out of Darcy's range and into the nonlinear range, then as n_1 decreases, n_2 increases. This transition is not well established experimentally. Finally equation (9) is valid for laminar flow when the order of R is above 10^{-3} . The ratio D_1/D_2 depends upon the absolute magnitude of the seepage velocity. Effect of non-homogeneity of the media and density is not discussed³.

Adsorption of cation in soil column

Cleary and Adrian² used one dimensional ADE to study the flow of cation solution through soils where linear equilibrium exists between cation in the flowing solution and the cation adsorbed on the exchanger phase. The equation considered in this case is

$$\frac{\partial C}{\partial t} + V_0 \frac{\partial C}{\partial Z} = D_0 \frac{\partial^2 C}{\partial Z^2} - \frac{\rho}{\epsilon} \frac{\partial S}{\partial t} \quad \dots (11)$$

Here, C is concentration of the adsorbate, V_0 is the average velocity, Z is longitudinal dimension along the direction of flow, D_0 is dispersion coefficient, ρ is bulk density, ϵ is pore fraction and S is the amount of adsorbate adsorbed per unit weight of the exchanger. One more term $\left(\frac{\rho}{\epsilon} \frac{\partial S}{\partial t}\right)$ is used to indicate the effect of density and pore fraction. Substituting $X = C/C_0$ and $Y = S/Q$ equation (11) is reduced into:

$$\frac{\partial X}{\partial t} + V \frac{\partial X}{\partial Z} = D \frac{\partial^2 X}{\partial Z^2} \quad \dots (12)$$

where,

$$D = \frac{D_0}{[1+(Q\rho/C_0\epsilon)]} \text{ and } V = \frac{V_0}{[1+(Q\rho/C_0\epsilon)]} \quad \dots (13)$$

and the boundary conditions are $X = 1$ at $Z = 0$, $\partial X/\partial Z = 0$ at $Z = L$ and $X = 0$ at $t = 0$ in $0 \leq Z \leq L$. Equation (12) is solved by using integral transform method after reducing it into diffusion equation and the analytical solution is expressed in terms of eigenvalues which is

$$X = 1 - 2 \sum_{m=1}^{\infty} \exp\left[(VZ/2D) - \left(\frac{V^2 t}{4D}\right) - D\beta_m^2 Z\right] \times \left[\frac{\beta_m^2 + \left(\frac{V}{2D}\right)^2}{L\left\{\beta_m^2 + \left(\frac{V}{2D}\right)^2\right\} + \frac{V}{2D}} \right] \text{Sin}(\beta_m Z) \times \exp\left(\frac{-VL}{2D}\right) \left[\frac{\left(\frac{V}{2D}\right) \text{Sin} \beta_m L - \beta_m \text{Cos} \beta_m L}{\left(\frac{V}{2D}\right)^2 + \beta_m^2} \right] + \beta_m \quad \dots (14)$$

where, the eigenvalues β_m 's are the positive roots of

$-\frac{V}{2D} \text{Sin} \beta_m L = \beta_m \text{Cos} \beta_m L$. This solution (14) is used to analyze the dispersion of cation in soil column. Using the data from Lai and Jurinak, cation concentration at different depth and time is shown in Figure 2, which shows that the depth covered by cation before reaching steady state is directly proportional to the time if concentration is kept constant².

Time dependent and spatial dependent dispersion

Atul Kumar et al.⁴ have studied one dimensional ADE, for two types of solute dispersion problems in a longitudinal finite domain. The equation taken is

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x} \left(D(x, t) \frac{\partial C}{\partial x} - u(x, t) C \right) \quad \dots (15)$$

where, C is concentration, D is dispersion, x spatial position, t is time and $u(x, t)$ is medium flow velocity. To indicate the change in dispersion with time we consider $D(x, t) = D_0 f(mt)$ and $u(x, t) = u_0$ where, D_0 is initial dispersion and u_0 is initial velocity.

The first case is time dependent solute dispersion along uniform flow and the second is spatial dependent solute dispersion along non-uniform flow. Both the problems contain uniform type of input and increasing nature of input. To indicate the change in dispersion and length a non-dimensional variable $f(mt)$ is taken where the unit of m is reciprocal of time.

$$f(mt) = \begin{cases} 1 & \text{for } m = 0 \text{ uniform solute dispersion} \\ 1 & \text{for } t = 0 \text{ initial solute dispersion} \end{cases} \quad \dots (16)$$

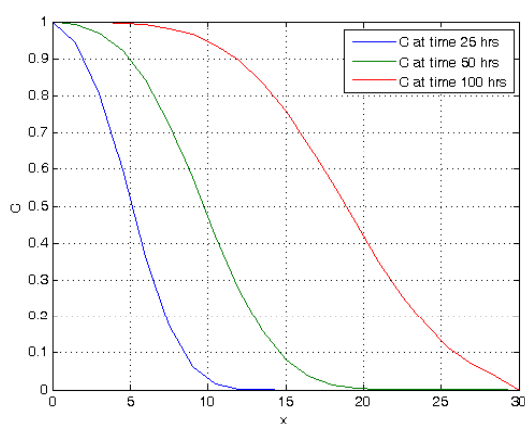


Figure 2: Cation concentration distribution for different time.

To use the Laplace transform technique conveniently a new independent variable X is introduced which is

$$X = \int \frac{dx}{f(mt)} \text{ or } \frac{dX}{dt} = \frac{1}{f(mt)} \quad \dots (17)$$

Again to get rid of the time dependent coefficient another transformation is used which is

$$T = \int_0^t \frac{dt}{f(mt)} \text{ or } \frac{dT}{dt} = \frac{1}{f(mt)} \quad \dots (18)$$

To vary the velocity by a small amount from u_0 at $x = 0$ to $u(x, t) = u_0(1 + b)$ at $x = L$ where, $b = ax$ for a real constant b is used. Similarly to represent the variation in dispersion $D(x, t) = D_0 (1 + ax)^2$ is used. Input concentration at the origin is assumed as C_0 and flux type homogeneous condition at the other end of the domain where $x = L$ is used. For spatially dependent dispersion along non-uniform flow a new variable is introduced for transformation.

$$X = - \int \frac{dx}{(1+ax)^2} = \frac{1}{a(1+ax)} \quad \dots (19)$$

Again to reduce into PDE with constant coefficient the variable used is $Z = -\log_a X = \log(1+ax)$. These functions we have discussed are used to reduce the ADE into standard form then after the solution for all cases is written directly from Van Genuchten and Alves¹⁰. For increasing and decreasing nature of input $f(mt) = \exp(mt)$ and $f(mt) = \exp(-mt)$ are used respectively⁴.

Distance up to 1 km, time up to 1 year, $C_0 = 1$, $u_0 = 0.11$ km/year and $D_0 = 0.21$ km²/year is taken. In the article it is explained that if the input is uniform, the concentration value is lower for $f(mt) = \exp(-mt)$ comparing with $f(mt) = \exp(mt)$ and the difference between two lines increases gradually in the domain. Again for increasing nature of input concentration up to 0.4 km is lower for $f(mt) = \exp(mt)$ and the trend reverses after that point. When the analytical and numerical values are compared, numerical value seems slightly higher near the source and trend reverses after 0.2 km approximately. The process of decaying of concentration is slower when dispersion is directly proportional to velocity and faster when the dispersion is proportional to the square of the velocity. The result is useful to predict the level of pollution at a particular point, to measure the water quality in subsurface areas and to calculate the atmospheric pollution⁴.

Analytical solution after breaking into two parts

Mojtabi and Deville⁵ solved one dimensional ADE with homogeneous boundary conditions which is

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = V \frac{\partial^2 u}{\partial x^2}, \quad -1 < x < 1, \quad t \in (0, T] \quad \dots (20)$$

In this case the finite domain is $-1 < x < 1$ and total time is T . Where, u is velocity variable, $c > 0$ is the constant advection velocity, v is the kinematic viscosity, t is time and the homogeneous Dirichlet boundary condition is $u(-1, t) = u(1, t) = 0$. The initial function chosen is $-\sin(\pi x)$ and the domain is $-1 < x < 1$. These boundary conditions are used already by Basdevant in Burger's equation¹. The ADE is first reduced into diffusion equation and solved by using variable separation method. The solution obtained in this

case contains the term $(0 \times \infty)$ for vanishing viscosity which makes the solution stiffer to calculate. Then variable separation method is left and the problem is decomposed into two parts advection and diffusion to solve them separately. After combining the final solution obtained is⁵

$$u(x,t) = \begin{cases} -\text{Sin}(\pi(1-ct)) \times \\ (1-v\pi^2(x+1)) \text{ for } ct < (x+1) \\ 0 \text{ if } ct > (x+1) \end{cases} \quad \dots (21)$$

Burgers equation: nonlinear advection diffusion equation

Sakai and Kimura⁷ solved Burger's equation, a fundamental nonlinear ADE which is formed after combining the nonlinear wave motion and linear diffusion term, which is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = v \frac{\partial^2 u}{\partial x^2} \quad \dots (22)$$

where, u is advective velocity and v is diffusivity. First the equation (22) is reduced into diffusion equation by using Cole-Hopf transformation which is $u = -2v \frac{\partial \theta / \partial x}{\theta}$

This transformation reduces the equation (22) into diffusion equation which is

$$\frac{\partial \theta}{\partial x} = v \left(\frac{\partial^2 \theta}{\partial x^2} \right) - \frac{1}{2v} \alpha(t) \theta$$

Then this equation is solved by using Fourier series method and the solution in the form of infinite exponential series under the sign of integration is

$$u(t,x) = v \times \int_a^b \sum_{k=-\infty}^{\infty} \left(\frac{x-x'-kL}{\mu} \right) \times e^{\left\{ \frac{-1}{2v} \int_a^{x'} u(t',x'') dx'' - \frac{(x-x'-kL)^2}{4\mu} \right\}} dx' \times \int_a^b \sum_{k=-\infty}^{\infty} e^{\left\{ \frac{-1}{2v} \int_a^{x'} u(t',x'') dx'' - \frac{(x-x'+kL)^2}{4\mu} \right\}} dx' \quad \dots (23)$$

Besides usual notations t' and t are old and new time respectively, v is viscosity and $\mu = \int_{t'}^t v dt$. The major part of this article is that the mathematical formulation of two dimensional Cole-Hopf transformation is described in detail. Then after two-dimensional nonlinear ADE is solved similarly as in case of one dimension⁷.

Conclusion

The articles we reviewed include both linear and nonlinear types of ADE. All analytical solution are obtained by reducing linear ADEs to diffusion equations except Ogata and Banks. The solution by Ogata and Banks⁶, expressed in terms of error function consists of two terms in which the first term is relevant for symmetrical boundary conditions. Inclusion of second term gives error up to five percent for general type of problems. Harleman and Rumer³ have explained a method to calculate Reynolds number by using the ratio of longitudinal and lateral dispersion. The result explained that the ratio of dispersion increases with increase in seepage velocity for Reynolds number less than 10 and not experimentally justified for greater than 10. Cleary and Adrian² have used the result to explain the cation concentration at different time and depth and concluded that if input is given for a longer time, the cation travel more distance before reaching the steady state. Atul Kumar et al.⁴ have used two suitable functions $\exp(mt)$ and $\exp(-mt)$ to indicate small amount of change which is better to study ADE. Abdelkadir and Michel⁵ have explained the idea to solve ADE after breaking into two parts, which gives similar result as Fourier solution. Sakai and Kimura⁷ have used the Cole-Hopf transformation to reduce the nonlinear ADE into diffusion equation and solved the equation in one and two dimension.

References

1. Basdevant, et al. 1986. Spectral and finite difference solutions of the burgers equation. *Journal of Computational Fluids*. **14**: 23-41.
2. Cleary, R.W. and Adrian, D.D. 1972. Analytical solution of the convective-dispersion equation for cation adsorption in soils. *Soil Science Society of America Journal*. **37(2)**: 197-199.
3. Harleman, D. R. F. and Rumer, R. R. 1963. Longitudinal and lateral dispersion in an isotropic porous medium. *Journal of Fluid Mechanics*. **16**: 385-394.
4. Kumar, A., Jaiswal, D. K. and Kumar, N. 2009. Analytical solution of one-dimensional advection-diffusion equation with variable coefficients in a finite domain. *Journal of Earth System Science*. **118(5)**: 539-549.
5. Mojtabi, A. and Deville, M. 2016. One-dimensional linear advection-diffusion equation: analytical and finite element solution. HAL archives-ouvertes, HAL Id: hal-01331727.

6. Ogata, A. and Banks, R. B. 1961. A solution of the differential equation of longitudinal dispersion in porous media. *Geological Survey Professional Papers*. 411-A, United States Government Printing Office, Washington.
7. Sakai, K. and Kimura, I. 2005. A numerical scheme based on a solution of nonlinear advection-diffusion equations. *Journal of Computational and Applied mathematics*. **173**: 39-55. doi:10.1016/j.cam.2004.02.019.
8. Stoker, T. 2016. *Introduction to Climate Modeling*. Public university in Bern Switzerland.
9. Tyn, M. U. and Lokenath, D. 2007. *Linear Partial Differential Equations for Scientists and Engineers*. 4th Edition, Birkhauser Publication Boston, eISBN-10:0-8176-4560-8.
10. Genuchten, M. T. V. and Alves, W. J. 1982. Analytical solution of one dimensional convective dispersive solute transport equations. *United States Department of Agriculture, Agriculture Research service*. **1661**: 151.