Further studying the Dirac field in FLRW space

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Abstract: Behaviour of the Dirac field in FLRW space is further investigated extending the previous work. WKB approximation up to the third order was used to solve the temporal part of the Dirac's equation and hence the energy and particle current have been calculated to further details.

Keywords: FLRW space-time; Dirac field; WKB approximation; NP formalism.

Introduction

A lot of efforts have been made to study and explain the formation and evolution of the Universe[1-6]. In this context, study of the matter fields, especially Dirac field[7-11], has given interesting results. In our previous works[12], writing the Dirac equation for FLRW space-time in Neumann-Penrose (NP) formalism[13], it was solved separating the angular, radial and temporal parts. WKB approximation up to the first order was used for the temporal part to explain the evolution of the universe. In this work, it is further investigated by extending it to the third order. Energy and particle current have been calculated with further details.

Separation of the Dirac field equations

Since it is the extension of our previous works[12], some equations have been gathered here from the work for ready reference. The metric for FLRW space-time can be written as

$$ds^2 = a^2[d\eta^2 - dr^2 - S^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \quad \dots (1)$$

where a is the scale factor which depends on the conformal time η and related to the co-moving time t by $dt = a\, d\eta$, and

$$S = \frac{\sin\sqrt{k}r}{\sqrt{k}} = \begin{cases} \sin r, & k = 1 \text{ for closed} \\ r, & k = 0 \text{ for flat} \\ \sinh r, & k = -1 \text{ for open} \end{cases} \quad \dots (2)$$

The temporal part separated from Dirac field equations written in NP-formalism came to be

$$\frac{1}{T_\pm}\left(\frac{\partial}{\partial\eta} \pm iMa\right)T_\pm = ik \quad \dots (3)$$

The separation constant is related to the co-moving momentum: $k = pa$

The decoupled equations are of second order and are given by

$$\left(\frac{\partial}{\partial\eta} \pm iMa\right)\left(\frac{\partial}{\partial\eta} \mp iMa\right)T_\pm = -k^2 T_\pm \quad \dots (4)$$

This can further be written as

$$\left(\frac{\partial^2}{\partial\eta^2} + k^2 + M^2a^2 \mp iMa'\right)T_\pm = 0 \quad \dots (5)$$

where

$$a'^2 = \left(\frac{da}{d\eta}\right)^2 = H_0^2[\Omega_\Lambda a^2 + (1 - \Omega_\Lambda - \Omega_M - \Omega_R)a^2 + \Omega_M a + \Omega_R] \quad \dots (6)$$

is given by the Friedmann-Lemaitre equation such that Ω_Λ, Ω_M, and Ω_R are densities contributed by the cosmological constant, matter and radiation, respectively, in units of the

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critical density at the time η_0 when the scale factor is normalized to $a_0 = 1$.

Solutions: energy and particle current

We assume that the expansion rate of the Universe is small compared to the size of the Universe, i.e., a changes very slowly with η so that $a(\eta)$ is small. With this assumption, we use the WKB approximation up to the third order to solve the eq. (5). For this, we multiply the derivative in eq. (5) with δ to incorporate the order of the approximation. Then substituting

$$T_\pm = exp\left(-\frac{i}{\delta}\sum_{n=0}^{\infty}\delta^n\int f_n d\eta\right) \quad \dots (7)$$

in the eq. (5), we get the equation for f as

$$\sum_{n=0}^{\infty}\delta^n\{\sum_{m=0}^{n} f_m f_{n-m} + i\, f'_{n-1}\} = k^2 + M^2a^2 \mp iMa' \dots (8)$$

Equating the coefficients of equal powers of δ, equations for the zeroth to the third orders of approximations are given by

$$f_0 = \sqrt{k^2+M^2a^2} = a\varepsilon_0 \quad \dots (9)$$

$$2f_0f_1 = -i(f'_0 \pm Ma') \quad \dots (10)$$

$$f_1^2 + 2f_0f_2 + if'_1 = 0 \quad \dots (11)$$

$$2f_0f_3 + 2f_1f_2 + if'_2 = 0 \quad \dots (12)$$

Solving these equations, expressions for various orders of f are obtained:

$$f_1 = -i\left(\frac{f'_0}{f_0} \pm \frac{Ma'}{f_0}\right) = -i\left(\frac{M^2}{\varepsilon_0^2} \pm \frac{M}{\varepsilon_0}\right)\frac{a'}{a} \dots (13)$$

$$f_2 = -i\frac{f'_1}{f_0} - \frac{f_1^2}{f_0} \quad \dots (14)$$

$$f_3 = -\frac{i}{2}\frac{f'_2}{f_0} - \frac{2f_1f_2}{f_0} \quad \dots (15)$$

Putting n = 0, 1, 2 and 3 in eq. (7) and then using equations (9) – (11), we get $T_\pm$ as functions of a, which would then lead to calculate the rate of energy flow and particle current. In order to express a', a'', and a''' in terms of a, eq. (6) can be used.

Some typical plots of them have been shown in Figure 1 and Figure 2. These are the plots for closed universe, *i.e.* K = 1 at present time (a = 1) and we have used the currently available values of H_0^2, Ω_Λ, Ω_M, and Ω_R.

While computing the energy and particle numbers, we have kept the terms up to the order of f_0'', or $f_0'^2$ *i.e.*, a_0'' or $a_0'^2$ or the similar ones, neglecting the higher order terms.

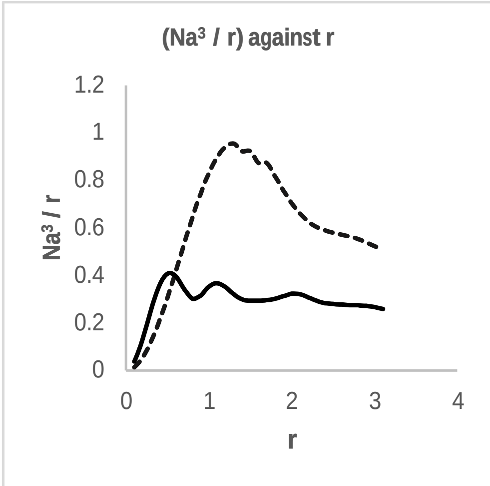

Figure 1: The number of Dirac particles (the local surface densities) divided by r in closed FLRW space-time. The dotted curve is for *k* = 3/2; *l* = 1/2, and the solid curve is for *k* = 7/2; *l* = 1/2. It is seen that as the Universe expands, the density contrast become more enhanced. It is also seen that the particle distribution has the same behaviour as that demanded by the flat rotation curves of galaxies.

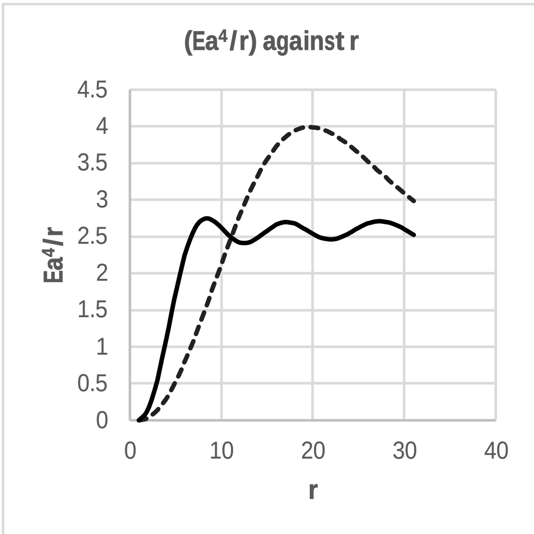

Figure 2: The energy of the Dirac field divided by r in closed FLRW space-time. The dotted curve is for k = 3/2; *l* = 1/2, and the solid curve is for k= 7/2; *l* = 1/2. The energy follows the particle number as it should.

Conclusions

In this work, we have further investigated the behaviour of the Dirac field in the closed FLRW space-time, particularly the temporal part, to the greater details using WKB approximation up to the third order.

The co-moving energy and the particle current calculated are dependent on $T_{\pm}$. Out of the solutions $|T_{+}|^2$ and $|T_{-}|^2$, one grows with time, while the other decays, but the sum is constant keeping the total particle numbers (or the total energy) conserved. The particle current or the energy flow are set up such that the over-densities are further enhanced whereas the under-densities get reduced while the Universe undergoes expansion. The expressions for energy and particle number density and hence their graphs show that the particle distribution has the same behaviour as that of the flat rotation curves of galaxies.

The results obtained from the third order approximation showed some finer corrections to the first order, but the patterns were maintained further supporting the previous findings.

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