Nepal Journal of Science and Technology Vol. 14, No. 1 (2013) 167-176

Censored Negative Exponential Distribution as a Mixed Distribution and Derivation of Its Moments

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Abstract

Censored negative exponential distribution is treated as a mixed type distribution having two distinct types of components. These components give arise to continuous as well as discrete random variables. Moments (mean and variance) are derived for the doubly censored, right censored, and left censored negative exponential distributions (NEDs) along with separations of continuous and discrete components and their respective means and variances. Moments obtained for the censored NEDs are then compared to the corresponding values of the uncensored NEDs and the changes in the proportions of the moments due to censoring are examined and assessed. Plots of moments of the censored distributions including a three dimensional scatter plot are presented considering different hypothetical values at which censoring may occur. These distributions are widely applied in fitting and modeling failure time data in survival and reliability analyses.

Key words: censoring, exponential distribution, failure time data, mixed distribution, moments

Introduction

Usually probability distributions are either discrete or continuous types. Occasionally, a probability distribution can be a combination of discrete as well as continuous random variables. Such distributions are referred as mixed type distributions. A mixed distribution can be decomposed into two components with one component leading to discrete random variable and another leading to continuous random variable. Alternatively, discrete and continuous distributions can be combined together to form a mixed distribution (Shrestha 2011). Mixed distributions have been used in different areas such as modeling daily amount of rainfall data (Sushaila *et. al.* 2011), insurance claim data in actuarial statistics, etc. In addition, probability distribution of a censored random variable can be treated as the probability distribution of a mixed random variable. Consequently, moments of a censored random variable can be derived using the principle of mixed distribution. The paper specifically deals with censored negative exponential distribution (NED) as a mixed distribution since the distribution is widely used in fitting and modeling failure time data in survival and reliability analyses (Hogg & Tanis 2001). Moments, specifically, mean and variance are derived for doubly and singly censored negative exponential distributions and are compared with the corresponding values of the moments of the distributions without censoring. The true values of the mean and variance of a random variable like failure time are the values obtained for uncensored data. Censoring take place due to some unavoidable compulsions such as fixation of the study period, etc. (Leung *et al.* 1997). Consequently, computations of mean and variance for censored data are different from the corresponding computations for uncensored distribution. The paper has attempted to derive functional relationships between the parametric values of uncensored and censored data under the assumption of NED of failure time data and examines in what proportions do the values for the moments change due to censoring. Moreover, failure time data can be characterized by different types of censoring. In the paper, censoring only due to the fixation of the study period (lower and / or upper limits of the failure time) is considered and no other types of censoring are assumed to occur during the study period.

Methodology

The principles of mixed distribution are used to associate it with censored distribution as follows (UAH 2012). A mixed type random variable can be associated with the Bernoulli random variable as an indicator variable with parameter P_C such that $I = 1$ in case of success and X is distributed as a continuous random variable X_C. On the other hand $I = 0$ in case of failure then X is distributed as a discrete random variable X_D, $1 - P_C = P_D$. Let $F(x)$ be the distribution function (DF) of the mixed random variable, $F_D(x)$ is the DF of the corresponding discrete random variable (X_D) which is the discrete component of the mixed distribution, and $F_C(x)$ is the DF of continuous random variable (X_C) which is the continuous component of the mixed distribution. Let D_F be a nonempty set of discontinuities such that $P(X \in D_F) \in (0,1)$. And D_F^c is the complement of. Thus, and. and are positive constants both adding to unity. Also, and . It follows that

$$P(X \le x) = P(X \in D_F)P(X \le x | X \in D_F) + P(X \in D_F^c)P(X \le x | X \in D_F^c)$$

so that

$$P(X \le x | X \in D_F^c) = \frac{P(X \le x) - P(X \in D_F)P(X \le x | X \in D_F)}{P(X \in D_F^c)}$$

Thus, $$F_C(x) = \frac{F(x) - P_D F_D(x)}{P_C} \quad (1)$$

Also,

$$P(X_D = x) = P(X = x | X \in D_F) = \frac{P(X = x \cap X \in D_F)}{P(X \in D_F)}$$

$$= \frac{P(X = x)}{P(X \in D_F)}$$

Thus, $$P(X_D = x) = \begin{cases} 0 & \text{if } X \in D_F^c \\ \dfrac{P(X = x)}{P(X \in D_F)} & \text{if } X \in D_F \end{cases} \quad (2)$$

Mean and variance of a mixed random variable can be obtained by adding means and variances from discrete and continuous distributions, respectively (UIOWA 2004).

Results and Discussion

Results are presented in different sections as follows.

Doubly censored negative exponential distribution

For a doubly censored NED, a new variable Y is defined as follows.

$$Y = \begin{cases} X & \text{if } a \le X \le b \\ a & \text{if } X < a \\ b & \text{if } X > b \end{cases} \quad (3)$$

where a , b are nonnegative constants and X is distributed as negative exponential distribution with parameter λ.

The DF of the new variable is:

$$G(y) = \begin{cases} 0 & \text{if } y < a \\ 1 - e^{-\lambda y} & \text{if } a \le y < b \\ 1 & \text{if } y \ge b \end{cases} \quad (4)$$

so that $P(Y = a) = 1 - e^{-\lambda a}$ and $P(Y = b) = e^{-\lambda b}$. Consequently, the distribution is a mixed distribution since Y is discrete at a and b and continuous in the interval (a, b). Graph of the DF is shown below (Fig. 1) for $\lambda = 1$, , and . The graph depicts continuous curve between a and b which is due to continuous nature of the mixed variable in the range (a. b). At a, the distribution function jumps from 0 to $= 1 - e^{-\lambda a} = 0.3935$ and at b the distribution function jumps from $= 1 - e^{-\lambda b} = 0.8647$ to 1 which is due to the discrete nature of the variable at a and at b.

Mean and variance of the distribution are derived as follows.

Mean

Mean of the mixed negative exponential distribution can be computed by adding the mean from the discrete values (a and b) and mean from the interval (a, b) as follows.

Mean from discrete values is:

$$a \times P(Y = a) + b \times P(Y = b)$$
$$= a(1 - e^{-\lambda a}) + be^{-\lambda b}$$

Mean from interval is:

$$\int_a^b yf(y)dy = \lambda \int_a^b ye^{-\lambda y}dy$$

so that the mean of Y is:

$$E(Y) = a \times P(Y = a) + b \times P(Y = b) + \int_a^b yf(y)dy$$

$$= a(1 - e^{-\lambda a}) + be^{-\lambda b} + \lambda \int_a^b ye^{-\lambda y} dy$$

where $\int_a^b ye^{-\lambda y}dy$ is evaluated as follows. Let $u = y$ and $v = e^{-\lambda y}$ so that

$$\int_a^b uv\,dy = u\int v\,dy\Big|_a^b - \int_a^b \frac{du}{dy}\left(\int v\,dy\right)dy$$

$$\Rightarrow \int_a^b ye^{-\lambda y}dy = y\int e^{-\lambda y}dy\Big|_a^b - \int_a^b \frac{dy}{dy}\left(\int e^{-\lambda y}dy\right)dy = y \times \frac{e^{-\lambda y}}{(-\lambda)}\Big|_a^b - \int_a^b \frac{e^{-\lambda y}}{(-\lambda)}dy$$

$$= \frac{1}{\lambda^2}\left\{e^{-\lambda a}(a\lambda + 1) - e^{-\lambda b}(b\lambda + 1)\right\}$$

$$E(Y) = a(1 - e^{-\lambda a}) + be^{-\lambda b} + \frac{\lambda}{\lambda^2}\left\{e^{-\lambda a}(a\lambda + 1) - e^{-\lambda b}(b\lambda + 1)\right\}$$

$$= a + \frac{e^{-\lambda a}}{\lambda} - \frac{e^{-\lambda b}}{\lambda}$$

$$= \frac{1}{\lambda}\left(a\lambda + e^{-\lambda a} - e^{-\lambda b}\right)$$

Thus, mean of the doubly censored NED is $E(Y) = \frac{1}{\lambda}\left(a\lambda + e^{-\lambda a} - e^{-\lambda b}\right)$. (5)

Variance

As above, E (Y^2) is derived as follows.

where $\int_a^b y^2 e^{-\lambda y} dy$ is evaluated as follows. Let $u = y^2$ and so that

$$E(Y^2) = a^2 \times P(Y = a) + \int_a^b y^2 f(y)dy + b^2 \times P(Y = b)$$

$$= a^2(1 - e^{-\lambda a}) + \lambda\int_a^b y^2 e^{-\lambda y}dy + b^2 e^{-\lambda b}$$

$$\int_a^b uv\,dy = u\int v\,dy\Big|_a^b - \int_a^b \frac{du}{dy}\left(\int v\,dy\right)dy \Rightarrow \int_a^b y^2 e^{-\lambda y}dy = y^2\int e^{-\lambda y}dy\Big|_a^b - \int_a^b \frac{dy^2}{dy}\left(\int e^{-\lambda y}dy\right)dy$$

$$= \frac{1}{\lambda}\left(a^2 e^{-\lambda a} - b^2 e^{-\lambda b}\right) + \frac{2}{\lambda^2}\left\{e^{-\lambda a}(a\lambda + 1) - e^{-\lambda b}(b\lambda + 1)\right\}$$

$$E(Y^2) = a^2(1 - e^{-\lambda a}) + b^2 e^{-\lambda b} + a^2 e^{-\lambda a} - b^2 e^{-\lambda b} + \frac{2}{\lambda^2}\left\{e^{-\lambda a}(a\lambda + 1) - e^{-\lambda b}(b\lambda + 1)\right\}$$

$$= a^2 + \frac{2}{\lambda^2}\left\{(a\lambda + 1)e^{-\lambda a} - (b\lambda + 1)e^{-\lambda b}\right\}$$

Also,

$$\{E(Y)\}^2 = \left\{\frac{1}{\lambda}\left(a\lambda + e^{-\lambda a} - e^{-\lambda b}\right)\right\}^2 = a^2 + \frac{2a}{\lambda}\left(e^{-\lambda a} - e^{-\lambda b}\right) + \frac{1}{\lambda^2}\left(e^{-2\lambda a} - 2e^{-\lambda a}e^{-\lambda b} + e^{-2\lambda b}\right)$$

Then,

$$Var(Y) = E(Y)^2 - \{E(Y)\}^2$$

$$= a^2 + \frac{2}{\lambda^2}\left\{(a\lambda + 1)e^{-\lambda a} - (b\lambda + 1)e^{-\lambda b}\right\}$$

$$- a^2 - \frac{2a}{\lambda}\left(e^{-\lambda a} - e^{-\lambda b}\right) - \frac{1}{\lambda^2}\left(e^{-2\lambda a} - 2e^{-\lambda a}e^{-\lambda b} + e^{-2\lambda b}\right)$$

$$= \frac{2}{\lambda^2}\left(e^{-\lambda a} - e^{-\lambda b}\right) + \frac{2}{\lambda}(a - b)e^{-\lambda b} - \frac{1}{\lambda^2}\left(e^{-2\lambda a} - 2e^{-\lambda(a+b)} + e^{-2\lambda b}\right)$$

Thus, variance of the doubly censored negative exponential distribution is

$$Var(Y) = \frac{2}{\lambda^2}\left(e^{-\lambda a} - e^{-\lambda b}\right) + \frac{2}{\lambda}(a - b)e^{-\lambda b} - \frac{1}{\lambda^2}\left(e^{-2\lambda a} - 2e^{-\lambda(a+b)} + e^{-2\lambda b}\right). \quad (6)$$

Right Censored Negative Exponential Distribution

For right censored distribution, a new variable Y is defined as follows:

$$Y = \begin{cases} X & \text{if } X \le b \\ b & \text{if } X > b \end{cases} \quad (7)$$

The CDF of the new variable is:

$$G(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - e^{-\lambda y} & \text{if } 0 \le y < b \\ 1 & \text{if } y \ge b \end{cases} \quad (8)$$

Y is discrete at b so that $P(Y = b) = e^{-\lambda b}$ and continuous in the range [0, b). Graph of the CDF is shown below for $\lambda = 1$ and $b = 2$. Mean and variance of the right censored distribution are derived as follows.

Mean

$E(Y) = \frac{1}{\lambda}\left(a\lambda + e^{-\lambda a} - e^{-\lambda b}\right)$ is the mean of the doubly censored NED. For $a = 0$, mean is

$$E(Y) = \frac{1}{\lambda}\left(1 - e^{-\lambda b}\right). \quad (9)$$

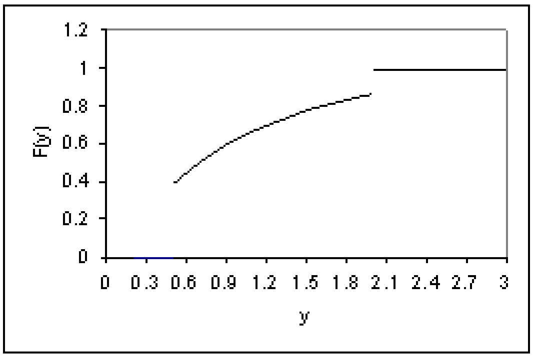

Fig. 1. Distribution function of doubly censored NED

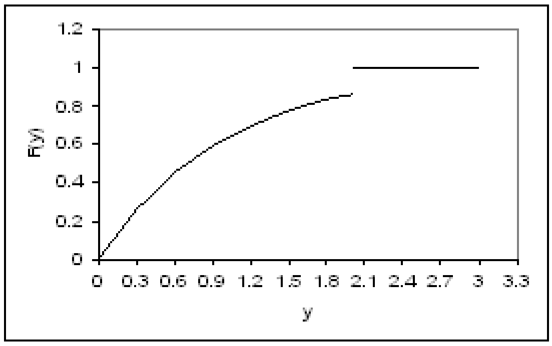

Fig. 2. Distribution function of right censored NED

Variance

Variance of the doubly censored NED is

$$Var(Y) = \frac{2}{\lambda^2}\left(e^{-\lambda a} - e^{-\lambda b}\right) + \frac{2}{\lambda}(a-b)e^{-\lambda b} - \frac{1}{\lambda^2}\left(e^{-2\lambda a} - 2e^{-\lambda(a+b)} + e^{-2\lambda b}\right)$$

For $a = 0$, variance is:

$$Var(Y) = \frac{2}{\lambda^2}\left(1 - e^{-\lambda b}\right) - \frac{2}{\lambda}be^{-\lambda b} - \frac{1}{\lambda^2}\left(1 - 2e^{-\lambda b} + e^{-2\lambda b}\right)$$
$$= \frac{1}{\lambda^2}\left(1 - e^{-2\lambda b}\right) - \frac{2}{\lambda}be^{-\lambda b} \quad (10)$$

Left censored negative exponential distribution

For left censored distribution, a new variable Y is defined as follows:

$$Y = \begin{cases} a & \text{if } X < a \\ X & \text{if } X \geq a \end{cases} \quad (11)$$

The CDF of the new variable is:

$$G(y) = \begin{cases} 0 & \text{if } y < a \\ 1 - e^{-\lambda y} & \text{if } y \geq a \end{cases} \quad (12)$$

Y is discrete at a so that $P(Y = a) = 1 - e^{-\lambda a}$ and continuous in the range (a, “). Mean and variance of the left censored distribution are derived as follows.

Mean

$E(Y) = \frac{1}{\lambda}\left(a\lambda + e^{-\lambda a} - e^{-\lambda b}\right)$ is the mean of the doubly censored NED. For b = “, mean is

$$E(Y) = \frac{1}{\lambda}\left(a\lambda + e^{-\lambda a}\right). \quad (13)$$

Variance

Variance of the doubly censored NED is:

$$Var(Y) = \frac{2}{\lambda^2}\left(e^{-\lambda a} - e^{-\lambda b}\right) + \frac{2}{\lambda}(a-b)e^{-\lambda b} - \frac{1}{\lambda^2}\left(e^{-2\lambda a} - 2e^{-\lambda(a+b)} + e^{-2\lambda b}\right)$$

For b = “, variance is:

$$Var(Y) = \frac{2}{\lambda^2}\left(e^{-\lambda a}\right) - \frac{1}{\lambda^2}\left(e^{-2\lambda a}\right) = \frac{1}{\lambda^2}\left(2e^{-\lambda a} - e^{-2\lambda a}\right) \quad (14)$$

Decomposition of the doubly censored negative exponential distribution

Mixed distribution of doubly censored NED can be decomposed into discrete and continuous components assuming the two components as events of a Bernoulli trial with success corresponded to continuous random variable and failure corresponded to discrete random variable. If P_C is the probability of the mixed random variable being continuous type and P_D is the probability of the mixed random variable being discrete type (Fig. 2) then $P_C + P_D = 1$. Consequently,

$$P_D = P(Y=a)+P(Y=b)=1-e^{-\lambda a}+e^{-\lambda b}$$

so that $P_C = 1-\left(1-e^{-\lambda a}+e^{-\lambda b}\right)=e^{-\lambda a}-e^{-\lambda b}$.

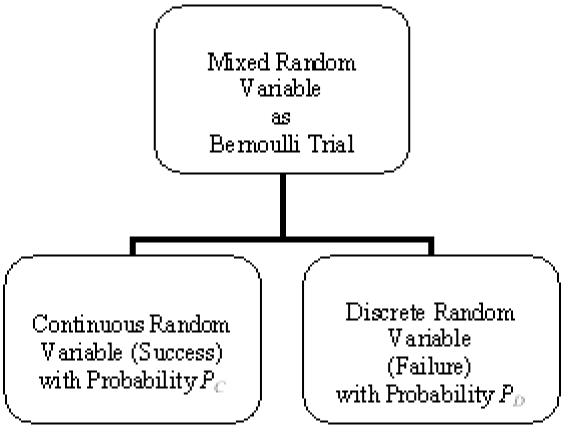

Fig. 3. M*ix*ed random variable corresponded to Bernoulli trial

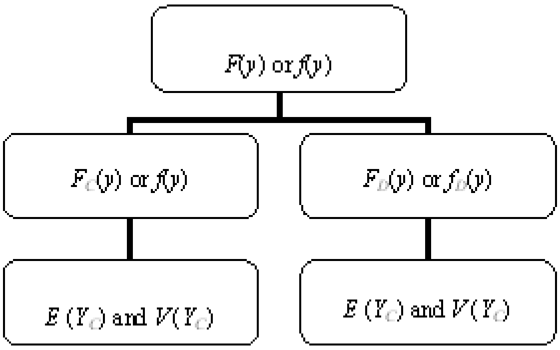

Fig. 4. Components of a mixed random variable

Discrete Component

PMF of Y_D is:

$$P(Y_D = y) = \begin{cases} 0 & \text{if } Y \in D_Y^c \\ \dfrac{1-e^{-\lambda a}}{1-e^{-\lambda a}+e^{-\lambda b}} & \text{if } y = a \\ \dfrac{e^{-\lambda b}}{1-e^{-\lambda a}+e^{-\lambda b}} & \text{if } y = b \end{cases} \qquad (15)$$

DF of Y_D is:

$$G_D(y) = \begin{cases} 0 & \text{if } y < a \\ \dfrac{1-e^{-\lambda a}}{1-e^{-\lambda a}+e^{-\lambda b}} & \text{if } a \le y < b \\ 1 & \text{if } y \ge b \end{cases} \qquad (16)$$

Mean of the discrete component is:

$$\begin{aligned} E(Y_D) &= y_1 P(Y_D = y_1) + y_2 P(Y_D = y_2) \\ &= a\frac{1-e^{-\lambda a}}{1-e^{-\lambda a}+e^{-\lambda b}} + b\frac{e^{-\lambda b}}{1-e^{-\lambda a}+e^{-\lambda b}} \\ &= \frac{a\left(1-e^{-\lambda a}\right)+be^{-\lambda b}}{1-e^{-\lambda a}+e^{-\lambda b}} \end{aligned} \qquad (17)$$

Similarly,

$$\begin{aligned} E\left(Y_D^2\right) &= y_1^2 P(Y_D = y_1) + y_2^2 P(Y_D = y_2) \\ &= \frac{a^2\left(1-e^{-\lambda a}\right)+b^2 e^{-\lambda b}}{1-e^{-\lambda a}+e^{-\lambda b}} \end{aligned}$$

Variance of the discrete component is:

$$\begin{aligned} Var(Y_D) &= E\left(Y_D^2\right) - \{E(Y_D)\}^2 \\ &= \frac{a^2\left(1-e^{-\lambda a}\right)+b^2 e^{-\lambda b}}{1-e^{-\lambda a}+e^{-\lambda b}} - \left\{\frac{a\left(1-e^{-\lambda a}\right)+be^{-\lambda b}}{1-e^{-\lambda a}+e^{-\lambda b}}\right\}^2 \\ &= \frac{(a-b)^2\left(1-e^{-\lambda a}\right)e^{-\lambda b}}{\left(1-e^{-\lambda a}+e^{-\lambda b}\right)^2} \end{aligned} \qquad (18)$$

Continuous Component

CDF of continuous component, YC is obtained as follows.

$$G(y) = 1-e^{-\lambda y};\ G_D(y) = \frac{1-e^{-\lambda a}}{1-e^{-\lambda a}+e^{-\lambda b}},\ a \le y < b;$$

$$G(y) = 1;\ G_D(y) = 1,\ y \ge b$$

Using $G_C(y) = \dfrac{G(y) - P_D G_D(y)}{P_C}$,

$P_D = 1-e^{-\lambda a}+e^{-\lambda b}$ $P_C = e^{-\lambda a}-e^{-\lambda b}$,

we obtain the CDF of YC as follows.

In the interval, $a \le y < b$

$$G_C(y) = \frac{G(y) - P_D G_D(y)}{P_C}$$

$$= \frac{1 - e^{-\lambda y} - \left(1 - e^{-\lambda a} + e^{-\lambda b}\right)\dfrac{1 - e^{-\lambda a}}{1 - e^{-\lambda a} + e^{-\lambda b}}}{e^{-\lambda a} - e^{-\lambda b}}$$

$$= \frac{e^{-\lambda a} - e^{-\lambda y}}{e^{-\lambda a} - e^{-\lambda b}}$$

Thus, the CDF of the continuous component of the mixed distribution can be expressed as:

$$G_C(y) = \begin{cases} 0 & \text{if } y < a \\ \dfrac{e^{-\lambda a} - e^{-\lambda y}}{e^{-\lambda a} - e^{-\lambda b}} & \text{if } a \le y < b \\ 1 & \text{if } y \ge b \end{cases} \qquad (19)$$

$$g_C(y) = \frac{\partial G_C(y)}{\partial y} = \frac{\partial \dfrac{e^{-\lambda a} - e^{-\lambda y}}{e^{-\lambda a} - e^{-\lambda b}}}{\partial y} \qquad (20)$$

$$= \frac{\lambda e^{-\lambda y}}{e^{-\lambda a} - e^{-\lambda b}}$$

Mean of is:

$$E(Y_C) = \int_a^b y g_c(y) dy = \frac{\lambda}{\left(e^{-\lambda a} - e^{-\lambda b}\right)} \int_a^b y e^{-\lambda y} dy$$

$$= \frac{1}{\lambda\left(e^{-\lambda a} - e^{-\lambda b}\right)}\left\{e^{-\lambda a}(a\lambda + 1) - e^{-\lambda b}(b\lambda + 1)\right\} \qquad (21)$$

Also,

$$E\left(Y_C^2\right) = \int_a^b y^2 g_c(y) dy = \frac{\lambda}{\left(e^{-\lambda a} - e^{-\lambda b}\right)} \int_a^b y^2 e^{-\lambda y} dy$$

$$= \frac{1}{\left(e^{-\lambda a} - e^{-\lambda b}\right)} \times \left[a^2 e^{-\lambda a} - b^2 e^{-\lambda b} + \frac{2}{\lambda^2}\left\{e^{-\lambda a}(a\lambda + 1) - e^{-\lambda b}(b\lambda + 1)\right\}\right] \qquad (22)$$

$$Var(Y_c) = E\left(Y_c^2\right) - \{E(Y_c)\}^2$$

$$= \frac{1}{\left(e^{-\lambda a} - e^{-\lambda b}\right)} \times \left[a^2 e^{-\lambda a} - b^2 e^{-\lambda b} + \frac{2}{\lambda^2}\left\{e^{-\lambda a}(a\lambda + 1) - e^{-\lambda b}(b\lambda + 1)\right\}\right]$$

$$- \left[\frac{1}{\lambda\left(e^{-\lambda a} - e^{-\lambda b}\right)}\left\{e^{-\lambda a}(a\lambda + 1) - e^{-\lambda b}(b\lambda + 1)\right\}\right]^2$$

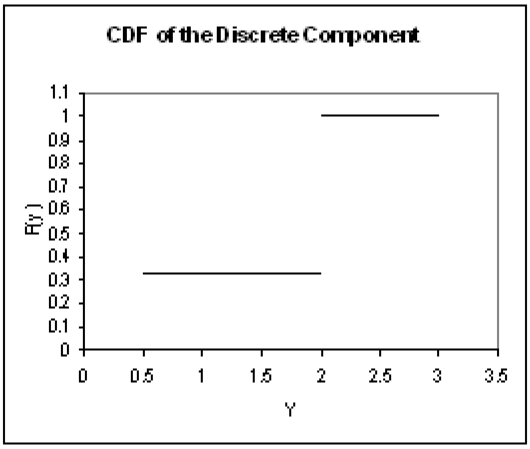

Fig. 5. CDF of discrete component

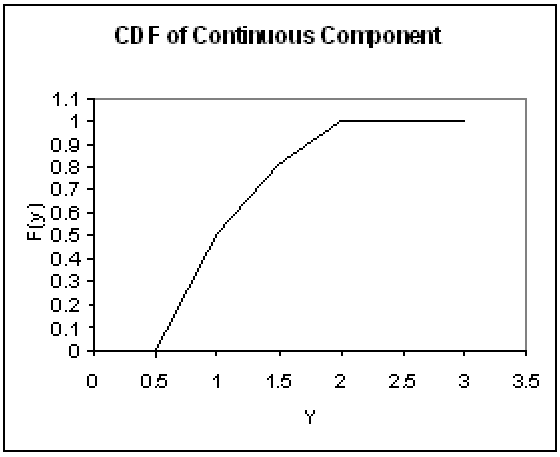

Fig. 6. CDF of continuous component

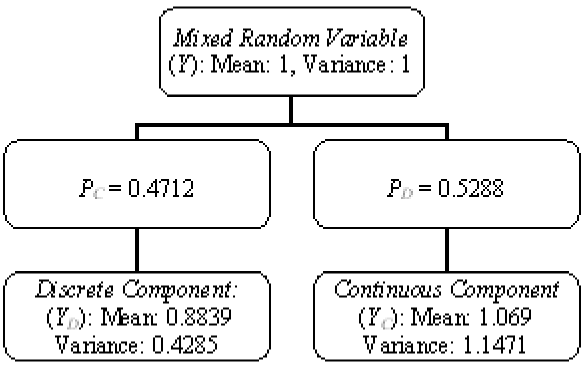

Fig. 7. Decomposition of mixed random variable

Decomposition of the right censored negative exponential distribution

The mixed distribution of right censored NED can be decomposed into discrete and continuous components as follows.

$P_D = P(Y = b) = e^{-\lambda b}$ so that $P_C = 1 - e^{-\lambda b}$.

Discrete Component

PMF of Y_D is: $P(Y_D = y) = \begin{cases} 0 \text{ if } Y \in D_F^c \\ 1 \text{ if } y = b \end{cases}$ (23)

DF of Y_D is: $G_D(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 & \text{if } y \geq 0 \end{cases}$ (24)

Mean of the discrete component is: $E(Y_D) = b$ (25)

ariance of the discrete component is: $Var(Y_D) = 0$ (26)

Continuous Component

CDF of continuous component, Y_C is obtained as follows.

$G(y) = 1 - e^{-\lambda y}$; $G_D(y) = 1$;

Using $G_C(y) = \dfrac{G(y) - P_D G_D(y)}{P_C}$, we obtain the CDF of as follows.

In the region, $y \geq 0$

$$G_C(y) = \frac{G(y) - P_D G_D(y)}{P_C} = \frac{1 - e^{-\lambda y} - \left(e^{-\lambda b}\right) \times 1}{1 - e^{-\lambda b}}$$

$$= 1 - \frac{e^{-\lambda y}}{1 - e^{-\lambda b}}$$

Thus, the CDF of the continuous component of the mixed distribution can be expressed as:

$$G_C(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 - \dfrac{e^{-\lambda y}}{1 - e^{-\lambda b}} & \text{if } 0 \leq y < b \\ 1 & \text{if } y \geq b \end{cases} \quad (27)$$

PDF of Y_C is:

$$g_C(y) = \frac{\partial G_C(y)}{\partial y} = \frac{\partial \left(1 - \dfrac{e^{-\lambda y}}{1 - e^{-\lambda b}}\right)}{\partial y}$$

$$= \frac{\lambda e^{-\lambda y}}{1 - e^{-\lambda b}}, \; 0 \leq y < b \quad (28)$$

Mean of Y_C is:

$$E(Y_C) = \int_0^b y g_c(y) dy = \frac{\lambda}{\left(1 - e^{-\lambda b}\right)} \int_a^b y e^{-\lambda y} dy$$

$$= \frac{1}{\lambda} - \frac{b e^{-\lambda b}}{1 - e^{-\lambda b}} \quad (29)$$

Also,

$$E\left(Y_C^2\right) = \int_a^b y^2 g_c(y) dy = \frac{\lambda}{\left(1 - e^{-\lambda b}\right)} \int_a^b y^2 e^{-\lambda y} dy$$

$$= \frac{2}{\lambda^2} - \frac{b^2 e^{-\lambda b}}{1 - e^{-\lambda b}} - \frac{2 b e^{-\lambda b}}{\lambda\left(1 - e^{-\lambda b}\right)}$$

$$Var(Y_C) = E\left(Y_C^2\right) - \{E(Y_C)\}^2$$

$$= \frac{2}{\lambda^2} - \frac{b^2 e^{-\lambda b}}{1 - e^{-\lambda b}} - \frac{2 b e^{-\lambda b}}{\lambda\left(1 - e^{-\lambda b}\right)} - \left[\frac{1}{\lambda} - \frac{b e^{-\lambda b}}{1 - e^{-\lambda b}}\right]^2$$

$$= \frac{1}{\lambda^2} - \frac{b^2 e^{-\lambda b}}{1 - e^{-\lambda b}} - \frac{b^2 e^{-2\lambda b}}{\left(1 - e^{-\lambda b}\right)^2} \quad (30)$$

It can be verified that combination of the discrete and continuous components give arise to the mixed distribution as follows.

$$E(Y) = P_D \times E(Y_D) + P_C \times E(Y_C)$$

$$= \left(e^{-\lambda b}\right) \times b + \left(1 - e^{-\lambda b}\right)\left(\frac{1}{\lambda} - \frac{b e^{-\lambda b}}{1 - e^{-\lambda b}}\right)$$

$$= \frac{1}{\lambda}\left(1 - e^{-\lambda b}\right)$$

The expression is identical with the mean of the mixed random variable as given in Equation 9.

Also,

$$Var(Y) = P_D \times Var(Y_D) + P_C \times Var(Y_C) + P_D P_C \{E(Y_D) - E(Y_C)\}^2$$

where

(a) $P_D \times Var(Y_D) = 0$

(b) $P_C \times Var(Y_C) = (1-e^{-\lambda b})\left(\frac{1}{\lambda^2} - \frac{b^2 e^{-\lambda b}}{1-e^{-\lambda b}} - \frac{b^2 e^{-2\lambda b}}{(1-e^{-\lambda b})^2}\right) = \frac{1-e^{-\lambda b}}{\lambda^2} - b^2 e^{-\lambda b} - \frac{b^2 e^{-2\lambda b}}{(1-e^{-\lambda b})}$

(c)

$$P_D P_C \left(E(Y_D) - E(Y_C)\right)^2 = e^{-\lambda b}(1-e^{-\lambda b})\left\{b - \left(\frac{1}{\lambda} - \frac{b e^{-\lambda b}}{1-e^{-\lambda b}}\right)\right\}^2 = \frac{e^{-\lambda b}}{\lambda^2 (1-e^{-\lambda b})}\left\{b\lambda - (1-e^{-\lambda b})\right\}^2$$

$$= \frac{b^2 e^{-\lambda b}}{(1-e^{-\lambda b})} - \frac{2b e^{-\lambda b}}{\lambda} + \frac{e^{-\lambda b}(1-e^{-\lambda b})}{\lambda^2}$$

Thus,

$$Var(Y) = \frac{1-e^{-\lambda b}}{\lambda^2} - b^2 e^{-\lambda b} - \frac{b^2 e^{-2\lambda b}}{(1-e^{-\lambda b})} + \frac{b^2 e^{-\lambda b}}{(1-e^{-\lambda b})} - \frac{2b e^{-\lambda b}}{\lambda} + \frac{e^{-\lambda b}(1-e^{-\lambda b})}{\lambda^2}$$

$$= \frac{(1-e^{-\lambda b} + e^{-\lambda b} - e^{-2\lambda b})}{\lambda^2} - b^2 e^{-\lambda b} + \frac{b^2 e^{-\lambda b}}{(1-e^{-\lambda b})}(1-e^{-\lambda b}) - \frac{2b e^{-\lambda b}}{\lambda}$$

$$= \frac{(1-e^{-2\lambda b})}{\lambda^2} - b^2 e^{-\lambda b} + b^2 e^{-\lambda b} - \frac{2b e^{-\lambda b}}{\lambda}$$

$$= \frac{(1-e^{-2\lambda b})}{\lambda^2} - \frac{2b e^{-\lambda b}}{\lambda}$$

The expression is identical with the variance of the mixed random variable as given in Equation 10.

Comparative assessment of the moments between censored and uncensored NEDs

The tables below displays values of the means and variances of the uncensored and censored NEDs, proportions of right censored NED means and variances compared to uncensored NED means and variances, respectively (for $\lambda = 0.2$). Tables demonstrate that as the time of right censoring increases the proportions of the means and variances of the right censored NED compared to uncensored NED also increases and approaches to unity for censoring times equivalent 40 years and 50 years, respectively if mean failure time is 5 years for uncensored NED (Table 2 & 3). The Tables also indicate that if the time of censoring is around the mean failure time then the mean and variance obtained from right censored distribution are much lower than the actual mean and variance of the uncensored NED. The values obtained in the tables can be utilized when estimates are obtained from right censored NED and we are interested to make assessment on the actual parameter values for the uncensored NED. The actual estimates for uncensored NED are likely to the values obtained by dividing the estimates obtained from right censored NED by the proportions shown in the Tables for different times at censoring.

Table 1. Moments of negative exponential distribution

Distribution Type	Mean	Variance
Uncensored	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Doubly Censored	$\frac{1}{\lambda}\left(a\lambda + e^{-\lambda a} - e^{-\lambda b}\right)$	$\frac{2}{\lambda^2}\left(e^{-\lambda a} - e^{-\lambda b}\right) + \frac{2}{\lambda}(a-b)e^{-\lambda b} - \frac{1}{\lambda^2}\left(e^{-2\lambda a} - 2e^{-\lambda(a+b)} + e^{-2\lambda b}\right)$
Right Censored	$\frac{1}{\lambda}\left(1 - e^{-\lambda b}\right)$	$\frac{1}{\lambda^2}\left(1 - e^{-2\lambda b}\right) - \frac{2}{\lambda}be^{-\lambda b}$
Left Censored	$\frac{1}{\lambda}\left(a\lambda + e^{-\lambda a}\right)$	$\frac{1}{\lambda^2}\left(2e^{-\lambda a} - e^{-2\lambda a}\right)$

Three dimensional scatter plot of mean in doubly censored NED

A typical three dimensional scatter plot of mean for different hypothetical values at which censoring can take place is plotted for an assumed value of mean in uncensored distribution. In the plot, *a* and *b* values are assigned in *X* and *Y* planes and means are assigned in the *Z* plane. The plot shows increasing values of the mean with increasing values of *a* and *b*.

Table 2. Proportion of right censored NED mean compared to uncensored NED Mean

Time at Censoring	Proportion
0	0.0000
5	0.6321
10	0.8647
15	0.9502
20	0.9817
25	0.9933
30	0.9975
35	0.9991
40	0.9997

Table 3. Proportion of right censored NED variance compared to uncensored NED Variance

Time at censoring	Proportion
0	0.0000
5	0.1289
10	0.4403
15	0.6988
20	0.8531
25	0.9326
30	0.9702
35	0.9872
40	0.9946
45	0.9978
50	0.9991

Proportion of right censored NED mean compared to uncensored NED mean

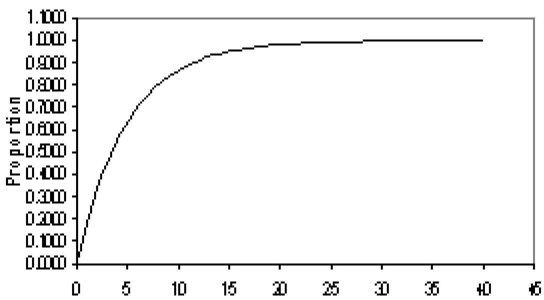

Fig. 8. Proportion of right censored NED mean

Proportion of right censored NED variance compared to uncensored NED variance

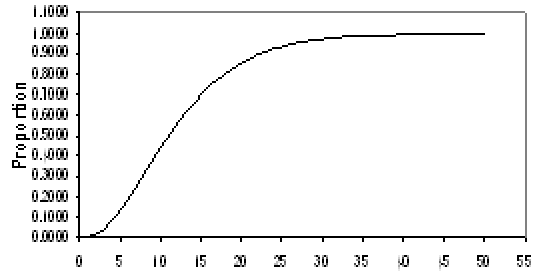

Fig. 9. Proportion of right censored NED variance

Mean of doubly censored negative exponential distribution

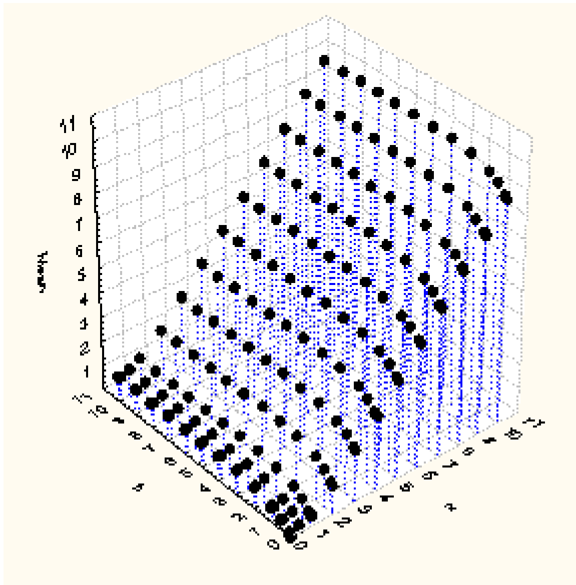

Fig. 10. Mean of doubly censored NED

Censored NED is treated as the mixed type probability distribution having two components with one component leading to a discrete random variable and another leading to a continuous random variable. Consequently, moments of the censored NED are derived using the principle of mixed distribution. Since NED is widely used as a parametric distribution in modeling varied types of data such as failure time data in survival and reliability analysis (Hosmer *et. al.* 2008), insurance claim data in actuarial statistics, warranty claim data in consumer product sales, etc., it is considered in this paper under the situations of censoring. Moments, specifically, mean and variance are derived for doubly and singly NEDs and are compared with the corresponding values of the moments of the NED without censoring. The values of the mean and variance of failure time are often observed along with censoring which leads inaccurate estimation of the true values of the moments if estimates applicable for only uncensored data are used. Thus, in the paper, functional relationships between the parametric values of uncensored and censored NEDs of failure time are derived and changes in proportions of the moments due to censoring are examined and assessed. For instance, as an illustration, the functional relationship shows that mean of failure time for = 0.2 is significantly less (proportion = 0.8647) than the true mean of uncensored NED if time at right censoring is 10 years and approaches to true mean if time at right censoring is about 30 years. Consequently, sample estimate of mean obtained from censored data is also likely to be less by a similar proportion as found for its parametric value. Similarly, the functional relationship shows that variance of failure time for λ = 0.2 is significantly less (proportion = 0.8531) than the true variance of uncensored NED if time at right censoring is 20 years and approaches to true variance if time at right censoring is about 50 years. Consequently, sample estimate the variance obtained from censored data is also likely to be less by a similar proportion as found for its parametric value. Moreover, failure time data can be characterized by different types of censoring. In the paper, censoring only due to the fixation of the study period (lower and upper limits of the failure time) is considered and no censoring occurs in between the study period.

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