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On Vector Valued Paranormed Sequence Space $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$ Defined by Orlicz Function

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Abstract

The aim of this paper is to introduce and study new classes $l_\infty(X, M, \bar{\lambda}, \bar{p})$ and $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$ of vector valued sequences by using Orlicz function M. We examine conditions pertaining the containment relation of the class $l_\infty(X, M, \bar{\lambda}, \bar{p})$ and explore the linear topological structure of vector valued sequence space $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$.

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Introduction:

Before proceeding with the main results we recall some terminology and notations. An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non decreasing and convex with $M(0) = 0$, $M(u) > 0$ for $u > 0$ and $M(u) \to \infty$ as $u \to \infty$. An Orlicz function satisfies the inequality $M(\alpha u) \le \alpha M(u)$ for all α satisfying $0 < \alpha < 1$. An Orlicz function M is said to satisfy Δ_2-condition for all values of $u \ge 0$, if there exists a constant $K > 0$ such that $M(2u) \le KM(u)$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(ru) \le K r M(u)$ for all values of u and for $r > 1$, (see, Krasnosel'skiĭ, M.A. *et al.* (1961)).

Lindentrauss and Tzafriri (1971) used the notion of Orlicz function to construct the Orlicz sequence space

$$l_M = \left\{ \bar{\eta} = (\eta_k) \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|\eta_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

of scalars ,which forms a Banach space with Luxemburg norm defined by

$$\| \bar{\eta} \|_M = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|\eta_k|}{\rho}\right) \le 1 \right\}.$$

Subsequently Parashar and Choudhary (1994) ,Ghosh and Srivastava (1999), Rao and Subramanian (2004),Tripathy *et al.* (2005), Karakaya(2005), Savas and Patterson(2005) , Khan (2008), Basariv and Altundag (2009) and many others have studied the algebraic and topological properties of sequence spaces defined by Orlicz functions.

Let X be a normed space over $\boldsymbol{C}$,the field of complex numbers. Let $\omega(X)$ denote the linear space of all sequences $\bar{x} = (x_k)$ with $x_k \in X$, $k \ge 1$ with usual coordinate wise operations i.e., $\bar{x} + \bar{y} = (x_k + y_k)$ and $\alpha\bar{x} = (\alpha x_k)$,for each $\bar{x}, \bar{y} \in \omega(X)$ and $\alpha \in \boldsymbol{C}$. We shall denote $\omega(\boldsymbol{C})$ by ω . Thus if $\bar{\lambda} = (\lambda_k) \in \omega$ and $\bar{x} \in \omega(X)$ then we shall write $\bar{\lambda}\,\bar{x} = (\lambda_k x_k)$.

The notion of paranormed spaces is closely related to linear metric spaces. (Wilansky (1978). A linear topological space X over $\boldsymbol{R}$ is said to be a paranormed space if there is a sub additive function $G : X \to \boldsymbol{R}$ (called paranorm on X) satisfying $G(0) = 0$,

$G(-x) = G(x)$ for all $x \in X$ and if (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ as $n \to \infty$ and (x_n) is a sequence of vectors with $G(x_n - x) \to 0$ as $n \to \infty$, then $G(\alpha_n x_n - \alpha x) \to 0$ as $n \to \infty$ (continuity of scalar multiplication). A paranorm G for which $G(x) = 0$ implies $x = 0$ is called **total**.

Note that the continuity of scalar multiplication is equivalent to

(i) if $G(x_n) \to 0$ and $\alpha_n \to \alpha$ as $n \to \infty$, then $G(\alpha_n x_n) \to 0$ as $n \to \infty$ and

(ii) if $\alpha_n \to 0$ as $n \to \infty$ and x be any element in X, then $G(\alpha_n x) \to 0$, (Wilansky 1978).

A sequence space S is said to be *normal* if $\bar{x} = (x_k) \in S$ and $\bar{\alpha} = (\alpha_k)$ a sequence of scalars with $|\alpha_k| \leq 1$, for all $k \geq 1$, then $\bar{\alpha}\bar{x} = (\alpha_k x_k) \in S$.

Following inequality has been used throughout this paper:

$|a + b|^n \leq |a|^n + |b|^n$, where $a, b \in \mathbf{C}$, $0 < n < 1$.

The Classes $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$ and $l_\infty(X, M, \bar{\lambda}, \bar{p})$ of Vector Sequences

Let $\bar{p} = (p_k)$ and $\bar{q} = (q_k)$ be any sequences of strictly positive real numbers and $\bar{\lambda} = (\lambda_k)$ and $\bar{\mu} = (\mu_k)$ be sequences of non zero complex numbers. Assume that

$$0 < l \leq \inf_k p_k \leq \sup_k p_k = L < \infty.$$

We now introduce the following classes of Banach space X- valued sequences

$$l_\infty(X, M, \bar{\lambda}, \bar{p}) = \left\{\bar{x} = (x_k) : x_k \in X \text{ and } M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho}\right) < \infty, \text{for some } \rho > 0\right\}$$

and $l_\infty(X, M, \bar{\lambda}, \bar{p}, L) = \{\bar{x} = (x_k) : x_k \in X$ and

$$M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) < \infty, \quad \text{for some } \rho > 0\}.$$

Clearly $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$ is a subset of $l_\infty(X, M, \bar{\lambda}, \bar{p})$. Further when $p_k = 1$ for all k, then $l_\infty(X, M, \bar{\lambda}, \bar{p})$ will be denoted by $l_\infty(X, M, \bar{\lambda})$ and when $\lambda_k = 1$ for all k, then $l_\infty(X, M, \bar{\lambda}, \bar{p})$ will be denoted by $l_\infty(X, M, \bar{p})$. If $p_k = \lambda_k = 1$ for all k, then the class $l_\infty(X, M, \bar{\lambda}, \bar{p})$ will be denoted by $l_\infty(X, M)$.

The Class $l_\infty(X, M, \bar{\lambda}, \bar{p})$

In this section, we investigate some inclusion relations between the classes $l_\infty(X, M, \bar{\lambda}, \bar{p})$ arising in terms of different $\bar{p}$ and $\bar{\lambda}$. Throughout, we shall denote

$$t_k = \left|\frac{\lambda_k}{\mu_k}\right|^{p_k}, \; r_k = \frac{1}{p_k}, \; s_k = \frac{1}{q_k}, \; k \geq 1.$$

Lemma 3.1: $l_\infty(X, M, \bar{\lambda}, \bar{p}) \subset l_\infty(X, M, \bar{\mu}, \bar{p})$ **if and only if** $\liminf_k t_k > 0$.

Proof:

For the sufficiency, assume that $\liminf_k t_k > 0$. Then there exists $m > 0$ such that $m|\mu_k|^{p_k} < |\lambda_k|^{p_k}$ for all sufficiently large values of k.

Let $\bar{x} = (x_k) \in l_\infty(X, M, \bar{\lambda}, \bar{p})$, then for some $\rho > 0$,

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho}\right) < \infty.$$

Now we choose $\rho_1 > 0$ such that $\rho \leq m\rho_1$. Since M is non decreasing, we have

$$\sup_k M\left(\frac{\|\mu_k x_k\|^{p_k}}{\rho_1}\right) = \sup_k M\left(\frac{|\mu_k|^{p_k}\|x_k\|^{p_k}}{\rho_1}\right)$$

$$\leq \sup_k M\left(\frac{|\lambda_k|^{p_k}\|x_k\|^{p_k}}{m\rho_1}\right)$$

$$\leq \sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho}\right) < \infty,$$

showing that $\bar{x} \in l_\infty(X, M, \bar{\mu}, \bar{p})$ and hence

$$l_\infty(X, M, \bar{\lambda}, \bar{p}) \subset l_\infty(X, M, \bar{\mu}, \bar{p}).$$

For the necessity, assume that

$l_\infty(X, M, \bar{\lambda}, \bar{p}) \subset l_\infty(X, M, \bar{\mu}, \bar{p})$ holds but $\liminf_k t_k = 0$.

So that we can find a sequence $(k(n))$ of integers such that $k(n+1) > k(n) \geq 1$, $n \geq 1$, satisfying

$$n|\lambda_{k(n)}|^{p_{k(n)}} < |\mu_{k(n)}|^{p_{k(n)}}, \text{ for all } n \geq 1.$$

Corresponding to $u \in X$ with $\|u\| = 1$, we define a sequence $\bar{x} = (x_k)$ by

$$x_k = \lambda_{k(n)}^{-1} u, \text{ for } k = k(n), \; n \geq 1$$
$$= 0, \text{ otherwise.}$$

Let $\rho > 0$. Then for $k = k(n)$, $n \geq 1$, using convexity of M we have

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho}\right) = \sup_n M\left(\frac{\|u\|^{p_{k(n)}}}{\rho}\right)$$

$$= M\left(\frac{1}{\rho}\right) < \infty,$$

$$\text{and } \sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho}\right) = 0, \text{ otherwise,}$$

showing that $\bar{x} \in l_\infty(X, M, \bar{\lambda}, \bar{p})$. But on the other hand, for any $\rho > 0$ and $k = k(n)$, $n \geq 1$, we have

$$\sup_k M\left(\frac{\|\mu_k x_k\|^{p_k}}{\rho}\right) = \sup_n M\left(\frac{\left\|\frac{\mu_{k(n)}}{\lambda_{k(n)}} u\right\|^{p_{k(n)}}}{\rho}\right) = \sup_n M\left(\left|\frac{\mu_{k(n)}}{\lambda_{k(n)}}\right|^{p_{k(n)}} \frac{1}{\rho}\right) \geq \sup_n M\left(\frac{n}{\rho}\right) = \infty,$$

showing that $\bar{x} \notin l_\infty(X, M, \bar{\mu}, \bar{p})$, a contradiction. This completes the proof.

Lemma 3.2: $l_\infty(X,M,\bar{\mu},\bar{p}) \subset l_\infty(X,M,\bar{\lambda},\bar{p})$ **if and only if** $\lim\sup_k t_k < \infty$.

Proof:
For the sufficiency, assume that $\lim\sup_k t_k < \infty$. Then we can find a positive number T such that
$T|\mu_k|^{p_k} > |\lambda_k|^{p_k}$ for all sufficiently large values of k.
Then analogous to the Lemma 3.1, the result follows.
For the necessity, suppose that
$l_\infty(X,M,\bar{\mu},\bar{p}) \subset l_\infty(X,M,\bar{\lambda},\bar{p})$ holds but $\lim\sup_k t_k = \infty$.
Then there exists a sequence $(k(n))$ of positive integers satisfying $k(n+1) > k(n) \geq 1$, $n \geq 1$, for which

$$\left|\frac{\lambda_{k(n)}}{\mu_{k(n)}}\right|^{p_{k(n)}} > n, \text{ for all } n \geq 1.$$

Now, corresponding to $u \in X$ with $\|u\| = 1$, define a sequence $\bar{x} = (x_k)$ by

$x_k = \mu_{k(n)}^{-1} u$, for $k = k(n)$, $n \geq 1$
$= 0$, otherwise.

Let $\rho > 0$. Then for $k = k(n)$, $n \geq 1$ and using convexity of M, we have

$$\sup_k M\left(\frac{\|\mu_k x_k\|^{p_k}}{\rho}\right) = \sup_n M\left(\frac{\|u\|^{p_{k(n)}}}{\rho}\right) = M\left(\frac{1}{\rho}\right) < \infty,$$

and $\sup_k M\left(\frac{\|\mu_k x_k\|^{p_k}}{\rho}\right) = 0$, otherwise,

which shows that $\bar{x} \in l_\infty(X, M, \bar{\mu}, \bar{p})$.
But on the other hand for any $\rho > 0$ and $k = k(n)$, $n \geq 1$, we have

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho}\right) = \sup_n M\left(\frac{\left\|\frac{\lambda_{k(n)}}{\mu_{k(n)}} u\right\|^{p_{k(n)}}}{\rho}\right) = \sup_n M\left(\left|\frac{\lambda_{k(n)}}{\mu_{k(n)}}\right|^{p_{k(n)}} \frac{1}{\rho}\right) \geq \sup_n M\left(\frac{n}{\rho}\right) = \infty,$$

showing that $\bar{x} \notin l_\infty(X, M, \bar{\lambda}, \bar{p})$, a contradiction. This completes the proof.
On combining the Lemmas 3.1 and 3.2, we get

Theorem 3.3: $l_\infty(X,M,\bar{\lambda},\bar{p}) = l_\infty(X,M,\bar{\mu},\bar{p})$ ***if and only if*** $0 < \lim\inf_k t_k < \lim\sup_k t_k < \infty$.

Corollary 3.4:

(i) $l_\infty(X,M,\bar{\lambda},\bar{p}) \subset l_\infty(X,M,\bar{p})$ ***if and only if*** $\lim\inf_k |\lambda_k|^{p_k} > 0$;

(ii) $l_\infty(X,M,\bar{p}) \subset l_\infty(X,M,\bar{\lambda},\bar{p})$ ***if and only if*** $\lim\sup_k |\lambda_k|^{p_k} < \infty$;

(iii) $l_\infty(X,M,\bar{\lambda},\bar{p}) = l_\infty(X,M,\bar{p})$ **if and only if** $0 < \lim\inf_k |\lambda_k|^{p_k} \leq \lim\sup_k |\lambda_k|^{p_k} < \infty$.

Proof:
By taking $\mu_k = 1$ for all k, in Lemmas 3.1, 3.2 and Theorem 3.3, the assertions (i), (ii) and (iii) follow.

Lemma 3.5: $l_\infty(X,M,\bar{\lambda},\bar{p}) \subset l_\infty(X,M,\bar{\lambda},\bar{q})$ **if and only if** $\lim\sup_k \frac{q_k}{p_k} < \infty$.

Proof:

For the sufficiency, assume that $\lim\sup_k \frac{q_k}{p_k} < \infty$. Then there exists $T > 0$ such that $q_k < T p_k$ for all sufficiently large values of k. Let $\bar{x} = (x_k) \in l_\infty(X,M,\bar{\lambda},\bar{p})$.

Then for some $\rho > 0$, $\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho}\right) < \infty$.

Hence we can find a real number $N > 1$ satisfying

$M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho}\right) < M\left(\frac{N}{\rho}\right)$, for all sufficiently large values of k. Since M is non decreasing, therefore $\|\lambda_k x_k\|^{p_k} \leq N$. This implies that $\|\lambda_k x_k\|^{q_k} \leq N^T$.

Hence, $\sup_k M\left(\frac{\|\lambda_k x_k\|^{q_k}}{\rho}\right) \leq M\left(\frac{N^T}{\rho}\right) < \infty$,

for all sufficiently large values of k and hence
$\bar{x} \in l_\infty(X,M,\bar{\lambda},\bar{q})$. Hence $l_\infty(X,M,\bar{\lambda},\bar{p}) \subset l_\infty(X,M,\bar{\lambda},\bar{q})$.
For the necessity, suppose that the inclusion holds but $\lim\sup_k \frac{q_k}{p_k} = \infty$. Then there exists a sequence $(k(n))$ of positive integers such that $k(n+1) > k(n) \geq 1$, $n \geq 1$, for which $q_{k(n)} > n p_{k(n)}$ for all $n \geq 1$.

Corresponding to $u \in X$ with $\|u\| = 1$, we define a sequence $\bar{x} = (x_k)$ by $x_k = \lambda_{k(n)}^{-1} 2^{n/q_{k(n)}} u$, for $k = k(n)$, $n \geq 1$ $= 0$, otherwise. So that for each $n \geq 1$, $k = k(n)$ and some $\rho > 0$, we have

$$\sup_k M\left(\frac{||\lambda_k x_k||^{p_k}}{\rho}\right) = \sup_n M\left(\frac{||2^{1/p_{k(n)}} u||^{p_{k(n)}}}{\rho}\right)$$
$$= \sup_n M\left(\frac{2||u||^{p_{k(n)}}}{\rho}\right)$$
$$= M\left(\frac{2}{\rho}\right) < \infty,$$

and $\sup_k M\left(\frac{||\lambda_k x_k||^{p_k}}{\rho}\right) = 0$, otherwise,

showing that $\bar{x} \in l_\infty(X, M, \bar{\lambda}, \bar{p})$. But for each $k = k(n)$, $n \geq 1$, we have

$$\sup_n M\left(\frac{||\lambda_k x_k||^{q_k}}{\rho}\right) = \sup_n M\left(\frac{||2^{1/p_{k(n)}} u||^{q_{k(n)}}}{\rho}\right)$$

Since, $q_{k(n)} / p_{k(n)} > n$ i.e. $2^{q_{k(n)}/p_{k(n)}} > 2^n$.

Since M is non decreasing, we have

$$\sup_k M\left(\frac{||\lambda_k x_k||^{q_k}}{\rho}\right) \geq \sup_n M\left(\frac{2^n ||u||^{q_{k(n)}}}{\rho}\right)$$
$$= \sup_n M\left(\frac{2^n}{\rho}\right) = \infty.$$

This shows that $\bar{x} \notin l_\infty(X, M, \bar{\lambda}, \bar{q})$, a contradiction. Hence the proof is complete.

Lemma 3.6: $l_\infty(X, M, \bar{\lambda}, \bar{q}) \subset l_\infty(X, M, \bar{\lambda}, \bar{p})$ *if and only if* $\liminf_k \frac{q_k}{p_k} > 0$.

Proof:

For the sufficiency, assume that $\liminf_k \frac{q_k}{p_k} > 0$. Then there exists a positive constant m such *that* $q_k > m\, p_k$, for all sufficiently large values of k.

Let $\bar{x} = (x_k) \in l_\infty(X, M, \bar{\lambda}, \bar{q})$. Then for some $\rho > 0$, $\sup_k M\left(\frac{||\lambda_k x_k||^{q_k}}{\rho}\right) < \infty$.

This shows that there exists a real number $N > 1$ satisfying

$M\left(\frac{||\lambda_k x_k||^{q_k}}{\rho}\right) < M\left(\frac{N}{\rho}\right)$, for all sufficiently large values of k. Since M is non decreasing, therefore $||\lambda_k x_k||^{q_k} < N$ and so $||\lambda_k x_k||^{p_k} < N^{1/m}$, for sufficiently large values of k. Hence using the convexity of M, we have

$$\sup_k M\left(\frac{||\lambda_k x_k||^{p_k}}{\rho}\right) \leq M\left(\frac{N^{1/m}}{\rho}\right) < \infty.$$

This implies that $\bar{x} \in l_\infty(X, M, \bar{\lambda}, \bar{p})$ and hence

$$l_\infty(X, M, \bar{\lambda}, \bar{q}) \subset l_\infty(X, M, \bar{\lambda}, \bar{p}).$$

For the necessity, assume that $l_\infty(X, M, \bar{\lambda}, \bar{q}) \subset l_\infty(X, M, \bar{\lambda}, \bar{p})$ but $\liminf_k \frac{q_k}{p_k} = 0$.

Then there exists a sequence $(k(n))$ of positive integers such that $k(n+1) > k(n) \geq 1$, for which $n\, q_{k(n)} < p_{k(n)}$ for each $n \geq 1$.

Corresponding to $u \in X$ with $||u|| = 1$, we define a sequence $\bar{x} = (x_k)$ by $x_k = \lambda_{k(n)}^{-1} 2^{1/q_{k(n)}} u$, for $k = k(n)$, $n \geq 1$ = 0, otherwise.

So that for each $n \geq 1$, $k = k(n)$ and some $\rho > 0$, we have

$$\sup_k M\left(\frac{||\lambda_k x_k||^{q_k}}{\rho}\right) = \sup_n M\left(\frac{||2^{1/q_{k(n)}} u||^{q_{k(n)}}}{\rho}\right)$$
$$= \sup_n M\left(\frac{2||u||^{q_{k(n)}}}{\rho}\right)$$
$$= M\left(\frac{2}{\rho}\right) < \infty,$$

and $\sup_k M\left(\frac{||\lambda_k x_k||^{q_k}}{\rho}\right) = 0$, for $k \neq k(n)$, $n \geq 1$,

showing that $\bar{x} \in l_\infty(X, M, \bar{\lambda}, \bar{q})$. But for each $k = k(n)$, $n \geq 1$, we have

$$\sup_k M\left(\frac{||\lambda_k x_k||^{p_k}}{\rho}\right) = \sup_n M\left(\frac{||2^{1/q_{k(n)}} u||^{p_{k(n)}}}{\rho}\right)$$
$$= \sup_n M\left(\frac{2^{p_{k(n)}/q_{k(n)}}}{\rho} ||u||^{p_{k(n)}}\right)$$
$$\geq \sup_n M\left(\frac{2^n}{\rho}\right) = \infty.$$

This shows that $\bar{x} \notin l_\infty(X, M, \bar{\lambda}, \bar{p})$, a contradiction. This completes the proof.

On combining the Lemmas 3.5 and 3.6, one obtain

Theorem 3.7: $l_\infty(X, M, \bar{\lambda}, \bar{p}) = l_\infty(X, M, \bar{\lambda}, \bar{q})$ *if and only if* $0 < \liminf_k \frac{q_k}{p_k} \leq \limsup_k \frac{q_k}{p_k} < \infty$.

Corollary 3.8:

(i) $l_\infty(X, M, \bar{\lambda}) \subset l_\infty(X, M, \bar{\lambda}, \bar{p})$ *if and only if* $\limsup_k p_k < \infty$;

(ii) $l_\infty(X, M, \bar{\lambda}, \bar{p}) \subset l_\infty(X, M, \bar{\lambda})$ **if and only if** $\liminf_k p_k > 0$;

(iii) $l_\infty(X, M, \bar{\lambda}, \bar{p}) = l_\infty(X, M, \bar{\lambda})$ *if and only if* $0 < \liminf_k p_k \leq \limsup_k p_k < \infty$.

Proof:

The proof follows by taking $p_k = 1$ for all k and $\bar{q}$ *is* replaced by $\bar{p}$ in the Lemmas 3.5 and 3.6 and Theorem 3.7.

Theorem 3.9: $l_\infty(X,M,\bar{\lambda},\bar{p}) \subset l_\infty(X,M,\bar{\mu},\bar{q})$ **if and only if**

(i) $\lim\inf_k t_k > 0$ **and (ii)** $\lim\sup_k \frac{q_k}{p_k} < \infty$.

Proof:
Proof of the theorem follows immediately from the Lemmas 3.1 and 3.5.

In the following example, we show that $l_\infty(X,M,\bar{\lambda},\bar{p})$ may strictly be contained in $l_\infty(X,M,\bar{\mu},\bar{q})$ in spite of the conditions (i) and (ii) of Theorem 3.9 are satisfied.

Example 3.10

Let X be a Banach space and consider a sequence $\bar{x} = (x_k)$ in X. Consider $u \in X$ such that $\|u\| = 1$ and define $x_k = k^k u$, if $k = 1, 2, 3, \ldots$

Further, let $p_k = k^{-1}$, if k is odd integer, $p_k = k^{-2}$, if k is even integer, $q_k = k^{-2}$ for all values of k, $\lambda_k = 3^k$, $\mu_k = 2^k$ for all values of k.

Then $t_k = \left|\frac{\lambda_k}{\mu_k}\right|^{p_k} = \frac{3}{2}$ or $\left(\frac{3}{2}\right)^{1/k}$ according as k is odd or even integer and hence $\lim\inf_k t_k = 1 > 0$. Further, $\frac{q_k}{p_k} = \frac{1}{k}$ if k is odd integer, $\frac{q_k}{p_k} = 1$, if k is even integer.

Therefore $\lim\sup_k \frac{q_k}{p_k} = 1 < \infty$. Hence the conditions (i) and (ii) of Theorem 3.9 are satisfied. Now, for some $\rho > 0$, we have

$$\sup_k M\left(\frac{\|\mu_k x_k\|^{q_k}}{\rho}\right) = \sup_k M\left(\frac{\|2^k k^k u\|^{1/k^2}}{\rho}\right) = \sup_k M\left(\frac{(2k)^{1/k}}{\rho}\|u\|^{1/k^2}\right) \le \sup_k M\left(\frac{(2k)^{1/k}}{\rho}\right) < \infty,$$

showing that $\bar{x} \in l_\infty(X,M,\bar{\mu},\bar{q})$. But for k an odd integer,

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho}\right) = \sup_k M\left(\frac{\|3^k k^k u\|^{1/k}}{\rho}\right) = \sup_k M\left(\frac{3k}{\rho}\right) = \infty.$$

This implies that $\bar{x} \notin l_\infty(X,M,\bar{\lambda},\bar{p})$. Thus, the containment of $l_\infty(X,M,\bar{\lambda},\bar{p})$ in $l_\infty(X,M,\bar{\mu},\bar{q})$ is strict inspite of the satisfaction of the conditions (i) and (ii) of the Theorem 3.9.

Linear Topological Structure of $l_\infty(X,M,\bar{\lambda},\bar{p},L)$

In this section, we shall investigate some theorems that characterize the linear topological structure of the space $l_\infty(X,M,\bar{\lambda},\bar{p},L)$ as defined earlier by endowing it a suitable paranorm.

Theorem 4.1: $l_\infty(X,M,\bar{\lambda},\bar{p})$ **forms a linear space over** C **if and only if** $\sup_k p_k < \infty$.

Proof:

Let $\bar{x}, \bar{y} \in l_\infty(X,M,\bar{\lambda},\bar{p})$ and $\alpha, \beta \in C$. Then there exist $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho_1}\right) < \infty \quad \text{and} \quad \sup_k M\left(\frac{\|\lambda_k y_k\|^{p_k}}{\rho_2}\right) < \infty.$$

Let us choose $\rho > 0$ satisfying
$2\rho_1 \max(1, |\alpha|) \le \rho$ and $2\rho_2 \max(1, |\beta|) \le \rho$.

For such ρ, using non decreasing and convex properties of M, we have

$$\sup_k M\left(\frac{\|\lambda_k(\alpha x_k + \beta y_k)\|^{p_k}}{\rho}\right)$$
$$\le \sup_k M\left(\frac{\|\alpha\lambda_k x_k\|^{p_k} + \|\beta\lambda_k y_k\|^{p_k}}{\rho}\right)$$
$$= \sup_k M\left(\frac{|\alpha|^{p_k}\|\lambda_k x_k\|^{p_k}}{\rho} + \frac{|\beta|^{p_k}\|\lambda_k y_k\|^{p_k}}{\rho}\right)$$
$$\le \sup_k M\left(\frac{1}{2\rho_1}\|\lambda_k x_k\|^{p_k} + \frac{1}{2\rho_2}\|\lambda_k y_k\|^{p_k}\right)$$
$$\le \frac{1}{2}\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k}}{\rho_1}\right) + \frac{1}{2}\sup_k M\left(\frac{\|\lambda_k y_k\|^{p_k}}{\rho_2}\right)$$
$$< \infty.$$

This implies that $l_\infty(X,M,\bar{\lambda},\bar{p})$ forms a linear space over C.

For the necessity, suppose that $l_\infty(X,M,\bar{\lambda},\bar{p})$ is a linear space over C but $\lim\sup_k p_k = \infty$. Then there exists a sequence $(k(n))$ of positive integers satisfying $k(n+1) > k(n) \ge 1$, $n \ge 1$, for which $p_{k(n)} > n$, for each $n \ge 1$.

Now, corresponding to $u \in X$ with $\|u\| = 1$, we define a sequence $\bar{x} = (x_k)$ by

$x_k = \lambda_{k(n)}^{-1} u$, for $k = k(n)$, $n \ge 1$
$= 0$, otherwise.

Then as in Theorem 3.2, we can show that

$\bar{x} \in l_\infty(X, M, \bar{\lambda}, \bar{p})$. On the other hand for any $\rho > 0$ and scalar $\beta = 4$, we get

$$M\left(\frac{\|\lambda_k \beta x_k\|^{p_k}}{\rho}\right) = M\left(\frac{\|4u\|^{p_{k(i)}}}{\rho}\right) \geq M\left(\frac{4^n}{\rho}\right) \geq M\left(\frac{4}{\rho}\right), \text{ for each } k \geq 1.$$

This shows that $\beta \bar{x} \notin l_\infty(X, M, \bar{\lambda}, \bar{p})$, a contradiction. This completes the proof.

Corollay. 4.2: *$l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$ forms a linear space over C.*

Proof:

Since by definition of $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$, L is finite and therefore by proceeding on the lines of proof of Theorem 4.1 the results follows.

In what follows for $\bar{x} \in l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$, we shall denote

$$\psi(\bar{x}) = \{\rho > 0 : \sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) \leq 1\}.$$

Theorem 4.3: *$l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$ forms a total paranormed space with respect to*

$$G(\bar{x}) = \inf\{\rho > 0 : \sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) \leq 1\}.$$

Proof:

Obviously, $G(0) = 0$ and $G(-\bar{x}) = G(\bar{x})$.

Further suppose that $G(\bar{x}) = 0$. Then for every $\varepsilon > 0$, there exists some ρ_ε $(0 < \rho_\varepsilon < \varepsilon)$, such that $\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho_\varepsilon}\right) \leq 1$. This shows that

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\varepsilon}\right) \leq 1, \text{ for every } \varepsilon > 0.$$

This is possible only when $\|\lambda_k x_k\|^{p_k/L} = 0$ for each $k \geq 1$. Hence $\bar{x} = 0$.

Now for $\bar{x}, \bar{y} \in l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$, consider $\rho_1 \in \psi(\bar{x})$ and $\rho_2 \in \psi(\bar{y})$. Then clearly by the convexity of M we have

$$M\left(\frac{\|\lambda_k(x_k + y_k)\|^{p_k/L}}{\rho_1 + \rho_2}\right)$$
$$\leq M\left[\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho_1} \times \frac{\rho_1}{\rho_1 + \rho_2} + \frac{\|\lambda_k y_k\|^{p_k/L}}{\rho_2} \times \frac{\rho_2}{\rho_1 + \rho_2}\right]$$
$$\leq \frac{\rho_1}{\rho_1 + \rho_2} \sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho_1}\right) + \frac{\rho_2}{\rho_1 + \rho_2} \sup_k M\left(\frac{\|\lambda_k y_k\|^{p_k/L}}{\rho_2}\right)$$
$$\leq \frac{\rho_1}{\rho_1 + \rho_2}.1 + \frac{\rho_2}{\rho_1 + \rho_2}.1 = 1.$$

This shows that $\rho_1 + \rho_2 \in \psi(\bar{x} + \bar{y})$.

Thus $G(\bar{x} + \bar{y}) \leq \rho_1 + \rho_2$ for each $\rho_1 \in \psi(\bar{x})$ and $\rho_2 \in \psi(\bar{y})$ implies that

$$G(\bar{x} + \bar{y}) \leq G(\bar{x}) + G(\bar{y}).$$

Finally we show the continuity of scalar multiplication. Let $\bar{x}^{(n)} = (x_k^{(n)})$ be a sequence in $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$ such that $G(\bar{x}^{(n)}) \to 0$ as $n \to \infty$ and (α_n) a sequence of scalars such that $\alpha_n \to \alpha$.

We prove that $G(\alpha_n \bar{x}^{(n)}) \to 0$.

$$G(\alpha_n \bar{x}^{(n)}) = \inf\left\{\rho : \sup_k M\left(\frac{\|\lambda_k \alpha_n x_k^{(n)}\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$
$$= \inf\left\{\rho : \sup_k M\left(\frac{|\alpha_n|^{p_k/L} \|\lambda_k x_k^{(n)}\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$
$$\leq \inf\left\{\rho : \sup_k M\left(\frac{H^{p_k/L} \|\lambda_k x_k^{(n)}\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$

where $H = \sup_n |\alpha_n|$. Thus for $t = \max(1, H)$, then we get

$$G(\alpha_n \bar{x}^{(n)}) \leq \inf\left\{\rho : \sup_k M\left(\frac{t\|\lambda_k x_k^{(n)}\|^{p_k/L}}{\rho}\right) \leq 1\right\}$$

Let $\frac{\rho}{t} = r$, so that

$$G(\alpha_n \bar{x}^{(n)}) \leq \inf\left\{rt : \sup_k M\left(\frac{\|\lambda_k x_k^{(n)}\|^{p_k/L}}{r}\right) \leq 1\right\}$$
$$= t \times P(\bar{x}^{(n)})$$

implies that $G(\alpha_n \bar{x}^{(n)}) \to 0$, as $G(\bar{x}^{(n)}) \to 0$ as $n \to \infty$.

Let $\alpha_n \to 0$ as $n \to \infty$ and $\bar{x}$ be any element in $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$. We show that $G(\alpha_n \bar{x}) \to 0$. Now for $0 < \varepsilon < 1$, we can find a positive integer N such that $|\alpha_n| \leq \varepsilon$ for all $n \geq N$. Since $\inf_k p_k = l > 0$, therefore $|\alpha_n|^{p_k/L} \leq |\alpha_n|^{l/L} \leq \varepsilon^{l/L}$ for all $n \geq N$.

So that

$$M\left(\frac{\|\alpha_n \lambda_k x_k\|^{p_k/L}}{\rho}\right) \leq M\left(\frac{|\alpha_n|^{p_k/L} \|\lambda_k x_k\|^{p_k/L}}{\rho}\right)$$

$$\leq \quad M\left(\frac{\varepsilon^{1/L} \| \lambda_k x_k\|^{p_k/L}}{\rho}\right).$$

For $\bar{x} \in l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$,

$$\psi(\bar{x}) = \{\rho > 0 : \sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) \leq 1\}.$$

So that $\psi(\varepsilon^{1/L}\bar{x}) = \{\rho > 0 : \sup_k M\left(\frac{\varepsilon^{1/L}\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) \leq 1\}$

and if $\sup_k M\left(\frac{\varepsilon^{1/L} \| \lambda_k x_k\|^{p_k/L}}{\rho}\right) \leq 1$, then

$$\sup_k M\left(\frac{\| \alpha_n \lambda_k x_k\|^{p_k/L}}{\rho}\right) \leq 1.$$

So, if $\rho \in \psi(\varepsilon^{1/L}\bar{x})$, then $\rho \in \psi(\alpha_n \bar{x})$

i.e., $\psi(\varepsilon^{1/L}\bar{x}) \subseteq \psi(\alpha_n \bar{x})$.

Taking infimum over such ρ's, we get

$$Inf\{\rho : \rho \in \psi(\alpha_n \bar{x})\} \leq Inf\{\rho : \rho \in A(\varepsilon^{1/L}\bar{x})\}$$

$$= \varepsilon^{1/L} \; Inf\{\rho : \rho \in \psi(\bar{x})\}$$

which shows that $G(\alpha_n \bar{x}) \leq \varepsilon^{1/L} G(\bar{x})$ for all $n \geq N$, i.e., $G(\alpha_n \bar{x}) \to 0$ as $n \to \infty$.

Hence $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$ forms a total paranormed space. This completes the poof.

Theorem 4.4: Total paranormed space $(l_\infty(X, M, \bar{\lambda}, \bar{p}, L), G)$ is complete.

Proof:

Let $(\bar{x}^{(i)})$ be a Cauchy sequence in $l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$. Let r be a fixed positive real number such that

$M(r) \geq 1$. Then for each $\frac{\varepsilon}{r} > 0$, there exists an integer $N \geq 1$ such that

$$G(\bar{x}^{(i)} - \bar{x}^{(j)}) < \frac{\varepsilon}{r} \text{ for all } i, j \geq N. \quad \ldots (4.1)$$

Using definition of paranorm, we see that

$$\sup_k M\left(\frac{\|\lambda_k x_k^{(i)} - \lambda_k x_k^{(j)}\|^{p_k/L}}{G(\bar{x}^{(i)} - \bar{x}^{(j)})}\right) \leq 1 \quad \ldots (4.2)$$

for all $i, j \geq N$.

Thus, $M\left(\frac{\|\lambda_k(x_k^{(i)} - x_k^{(j)})\|^{p_k/L}}{G(\bar{x}^{(i)} - \bar{x}^{(j)})}\right) \leq 1 \leq M(r)$, for all $i, j \geq N$ and $k \geq 1$.

But M is non decreasing, therefore

$$\frac{\|\lambda_k(x_k^{(i)} - x_k^{(j)})\|^{p_k/L}}{G(\bar{x}^{(i)} - \bar{x}^{(j)})} < r$$

Hence by using (4.1),

we have $\|\lambda_k(x_k^{(i)} - x_k^{(j)})\|^{p_k/L} < \varepsilon. \quad \ldots (4.3)$

This shows that $(x_k^{(i)})$ is a Cauchy sequence in X for all $k \geq 1$. But X is complete, therefore there exists x_k (say) in X for each $k \geq 1$ such that $x_k^{(i)} \to x_k$ as $i \to \infty$.

We show that $\bar{x} = (x_k) \in l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$.

Let us choose $\rho > 0$ such that

$P(\bar{x}^{(i)} - \bar{x}^{(j)}) < \rho < \varepsilon$ for all $i, j \geq N$. $\quad \ldots (4.4)$

Since M is non decreasing, therefore by (4.2) we have

$$\sup_k M\left(\frac{\|\lambda_k(x_k^{(i)} - x_k^{(j)})\|^{p_k/L}}{\rho}\right) \leq \sup_k M\left(\frac{\|\lambda_k(x_k^{(i)} - x_k^{(j)})\|^{p_k/L}}{G(\bar{x}^{(i)} - \bar{x}^{(j)})}\right)$$

$$\leq 1 \text{ for all } i, j \geq N.$$

Since M is continuous, taking limit as $j \to \infty$, we see that

$$\sup_k M\left(\frac{\|\lambda_k(x_k^{(i)} - x_k)\|^{p_k/L}}{\rho}\right) \leq 1 \text{ for all } i \geq N.$$

Taking infimum of such ρ's, we get

$$G(\bar{x}^{(i)} - \bar{x}) = \inf\{\rho : \sup_k M\left(\frac{\|\lambda_k(x_k^{(i)} - x_k)\|^{p_k/L}}{\rho}\right) \leq 1 \text{ for all } i \geq N\}$$

$$\leq \rho < \varepsilon.$$

$\Rightarrow G(\bar{x}^{(i)} - \bar{x}) < \varepsilon$, for all $i \geq N$.

This shows that $\bar{x}^{(i)} \to \bar{x}$ as $i \to \infty$ and clearly

$$\bar{x}^{(i)} - \bar{x} \in l_\infty(X, M, \bar{\lambda}, \bar{p}, L), \text{ for all } i \geq N.$$

Also, $\bar{x}^{(N)}$ and $\bar{x}^{(N)} - \bar{x} \in l_\infty(X, M, \bar{\lambda}, \bar{p}, L)$, therefore it follows that

$$\bar{x} = \bar{x}^{(N)} - (\bar{x}^{(N)} - \bar{x}) \in l_\infty(X, M, \bar{\lambda}, \bar{p}, L).$$

This completes the proof.

Theorem 4.5: The space $l_\infty(X,M,\bar{\lambda},\bar{p},L)$ is normal.

Proof:

Let $\bar{x} = (x_k) \in l_\infty(X,M,\bar{\lambda},\bar{p},L)$ .So that

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) < \infty \quad \text{for some } \rho > 0.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \le 1$ for all $k \ge 1$. Since M is non-decreasing, we have

$$\sup_k M\left(\frac{\|\lambda_k \alpha_k x_k\|^{p_k/L}}{\rho}\right) = \sup_k M\left(\frac{|\alpha_k|^{p_k/L}\|\lambda_k x_k\|^{p_k/L}}{\rho}\right)$$
$$\le \sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) < \infty,$$

and hence $(\alpha_k x_k) \in l_\infty(X,M,\bar{\lambda},\bar{p},L)$.

So $l_\infty(X,M,\bar{\lambda},\bar{p},L)$ is normal.

We now introduce a new sub class $\bar{l}_\infty(X,M,\bar{\lambda},\bar{p},L)$ of $l_\infty(X,M,\bar{\lambda},\bar{p},L)$ as follows:

$$\bar{l}_\infty(X,M,\bar{\lambda},\bar{p},L) = \{\bar{x} = (x_k) : x_k \in X,\ M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) < \infty \quad \text{for every } \rho > 0\}.$$

Theorem 4.6 If M satisfies Δ_2 condition then

$$l_\infty(X,M,\bar{\lambda},\bar{p},L) = \bar{l}_\infty(X,M,\bar{\lambda},\bar{p},L).$$

Proof:

To prove the theorem, it suffices to show that

$l_\infty(X,M,\bar{\lambda},\bar{p},L) \subseteq \bar{l}_\infty(X,M,\bar{\lambda},\bar{p},L)$.

Let $\bar{x} \in l_\infty(X,M,\bar{\lambda},\bar{p},L)$. Then for some $\rho > 0$,

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) < \infty.$$

Let us consider an arbitrary $\rho_1 > 0$.

Case I: If $\rho \le \rho_1$, then obviously we have

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho_1}\right) \le \sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) < \infty,$$

and hence we get $\bar{x} \in \bar{l}_\infty(X,M,\bar{\lambda},\bar{p},L)$.

Case II: If $\rho > \rho_1$, so that $\frac{\rho}{\rho_1} > 1$ then by using Δ_2-condition of M, we get

$$\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho_1}\right) = \sup_k M\left(\frac{\frac{\rho}{\rho_1}\|\lambda_k x_k\|^{p_k/L}}{\rho}\right)$$
$$\le K\frac{\rho}{\rho_1}\sup_k M\left(\frac{\|\lambda_k x_k\|^{p_k/L}}{\rho}\right) < \infty,$$

where K is the number involved in Δ_2- condition .

Hence $\bar{x} \in \bar{l}_\infty(X,M,\bar{\lambda},\bar{p},L)$.

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