Nepal Journal of Science and Technology 10 (2009) 149-153

Contraction Mapping Principle Approach to Differential Equations

Bishnu P. Dhungana
Department of Mathematics, Mahendra Ratna Campus
Tribhuvan University, Kathmandu Nepal

Abstract

Using an extension of the contraction mapping principle, a new approach has been proposed in proving the existence of unique solutions of some differential equations.

Key words: contraction, existence and uniqueness, differential equations.

Introduction

Over the past few decades there has been a clear emergence of the idea of contraction mapping in the realm of nonlinear functional analysis. With a germ of this idea, it was Banach who was first able to introduce and prove a very powerful principle, called *Contraction Mapping Principle*.

Beginning from the idea of a contraction T and its fixed point (Baily 1966, Rubinstein 1998, Yosida 1978) we state the principle and extend it to iterates i.e. a result similar to contraction mapping principle is obtained for a mapping T (not contraction) provided that some iterate of is a contraction. This paper is mainly concerned with the extraordinary applicability and effectiveness of the principle to evolve a number of useful results in differential equations. Particularly the strength will be given for a new approach in proving the existence of unique solutions of some differential equations with some initial conditions. Compared to the proofs through Picard iterates (Braun 1993) the method of our proof based on an extension of contraction mapping principle is a new approach. In fact the major technical novelty is to obtain the result (3.9) proving that is a contraction for some positive integer. Finally, some examples to illustrate the results are suitably provided.

Preliminaries

Many equations which are of interest in applications can be put in the form 0where is a mapping of some subset of a metric space into itself. Such a pointquite naturally is called a fixed point of . An ancient method of solving equations of the form is the method of iteration i.e. an initial approximationis chosen and successive approximationsare generated by the formula

$$x_n = T(x_{n-1}) \qquad (n = 1, 2, \cdots) \qquad (2.1)$$

If the mappingis continuous and if the sequence (x_n) converges to w, then

$$w = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T(\lim_{n\to\infty} x_{n-1}) = T(w)$$

Thus for continuous mapping T if the process (2.1) converges at all, then it converges to a fixed point of T. However, to prove the convergence of (2.1), we will in general need a condition on T which is much stronger than continuity. Specifically, we will require that T be contraction in the sense that it always maps any two points closer together in uniform way as expressed by the following definition.

Definition 2.1: Let (X, d) be a metric space. Then a contraction of (X, d) is a mapping $T : X \to X$ with the property that for some real number $k < 1$,

$$d(T(x), T(y)) \le k\, d(x, y) \qquad \forall x, y \in X.$$

Note that a differentiable mapping $T : [a, b] \to [a, b]$ is a contraction if and only if there is a number $k < 1$ with $|T'(x)| \le k \quad \forall x, y \in (a, b)$.

Theorem 2.1: (Contraction Mapping Principle, [2]) *Let $T : X \to X$ be a contraction of a complete metric space (X,d). Then T has a unique fixed point i.e. there exists a unique point w in X such that $T(w) = w$. Furthermore, if x_0 is any point of X and (x_n) is a sequence of iterates defined by $x_n = T(x_{n-1}), n = 1,2,\cdots$. Then0 $\lim_{n\to\infty} x_n = w$.*

Remark 2.1: Both conditions of Theorem 1.1 are necessary sincethe mapping $T:(0,1] \to (0,1]$ defined by $T(x) = x/2$ is a contraction map but has no fixed point since $(0,1]$ is not a complete metric space. the mapping $T: \boldsymbol{R} \to \boldsymbol{R}$ defined by $T(x) = x + 5$ is not a contraction and has no fixed point although $\boldsymbol{R}$ is complete.

If T is a contraction mapping, then T^n where n is a positive integer, is clearly a contraction mapping. However the converse may not be true as can be seen from the following example.

Example 2.1: The function $T : \boldsymbol{R} \to \boldsymbol{R}$ defined by $T(x) = e^{-x}$ is not a contraction, but that T^2 is.

Thus we see that provided some iterate of T is a contraction we still get a fixed point result similar to the contraction mapping principle for T. The following theorem is an extension of the principle to iterates.

Theorem 2.2: *Let (X,d) be a complete metric space and let $T : X \to X$ have the property that for some integer $N > 0$, the iterate T^N is a contraction of X. Then T has a unique fixed point i.e. there exists a unique point w in X such that $T(w) = w$. Furthermore, if x_1 is in X, then the sequence of iterates defined by $x_{n+1} = T(x_n), n \geq 1$ converges to w.*

Proof: Since T^N is a contraction of the complete metric space (X,d), it has a unique fixed point w, say. It then follows that

$$T(w) = T(T^N(w)) = T^{N+1}(w) = T^N(T(w)).$$

This shows that $T(w)$ is another fixed point of T^N. From uniqueness, T has a fixed point w in X and moreover it is unique since any point which T fixes clearly remains fixed by T^N. Finally to see that $x_1, x_2, x_3, \cdots$ converges to w, we re-label T^N as g and note that

$$x_{N+1} = T(x_N) = T(T(x_{N-1})) = \cdots = T^N(T(x_1)) = g(x_1)$$

$$x_{N+2} = T(x_{N+1}) = T(T(x_N)) = \cdots = T^N(T(x_2)) = g(x_2)$$

We now rewrite the sequence $x_1, x_2, x_3, \cdots$ as

$$x_1, x_2, x_3, \cdots; x_N, g(x_1), g(x_2), g(x_3), \cdots; g(x_N), g(g(x_1)), g(g(x_2)), g(g(x_3)); \cdots.$$

This is actually a combination of the N sequences

$$x_1, g(x_1), g(g(x_1)), \cdots$$
$$x_2, g(x_2), g(g(x\ \ )), \cdots$$
$$x_3, g(x_3), g(g(x_3)), \cdots$$
$$\vdots$$
$$x_N, g(x_N), g(g(x_N)), \cdots$$

Each row in the above 'array' is obtained by starting at some point of X and iterating with the contraction of g. By Theorem 1.1, any such sequence converges to a unique fixed point of $g = T^N$, namely w. Since each row in the above 'array' is a subsequence of the combined sequence $x_1, x_2, x_3, \cdots$ converging to w, the sequence must also converge to, as desired $\Diamond$.

Main results

Let F be a real valued function on a nonempty subset D of the Euclidean space $\boldsymbol{R}^2$. A real valued function ϕ on an interval I is said to be a solution of the differential equation

$$dx/dt = F(x,t) \qquad (3.1)$$

on the interval I if and only if $(\phi(t),t) \in D$ for all $t \in I$, is differentiable on I and $\phi'(t) = F(\phi(t),t)\ \forall t \in I$.

Definition 3.1: Let F be a continuous real valued function on a nonempty subset D of the Euclidean space R^2. A real valued function ϕ on an interval I containing c is said to be a solution of the integral equation

$$x(t) = x_0 + \int_c^t F(\phi(s), s)\,ds \quad (t \in I) \tag{3.2}$$

if and only if $(\phi(t), t) \in D$ for all $t \in I$, is continuous on I and for all $t \in I$

$$\phi(t) = x_0 + \int_c^t F(\phi(s), s)\,ds.$$

The integral $\int_c^t F(\phi(s), s)\,ds$ is defined for each $t \in I$ since the function $s \to F(\phi(s), s)$ is continuous on I. It is easy to see that the differential equation (3.1) with the initial condition $x(c) = x_0$ is equivalent to the integral equation (3.2).

Theorem 3.1: *Let $F : R \times [a, b] \to R$ be a function of two variables such that $F(x, t)$ is defined for all $x \in R$ and $t \in [a, b]$. Assume that F is continuous and that there exists a real number L with*

$$| F(x,t) - F(y,t) | \le L | x - y |$$

for all $x, y \in R$ and . Then the differential equation

$$dx / dt = F(x, t) \quad (3.3)$$

subject to an initial condition of the type $x(\alpha) = \beta$ has a unique solution.

Proof: Let $X = C[a,b]$ Then X is a complete metric space with the metric

$$d(x, y) = \sup_{a \le t \le b} | x(t) - y(t) | .$$

Define $T : X \to X$ by

$$(T(x))(t) = \beta + \int_a^t F(x(s), s)\,ds.$$

Then the fixed points of T are the solutions of the integral equation on $[a, b]$ and hence these are the solutions to the differential equation (3.3). To prove the theorem it suffices to show that T has a unique fixed point. We first show that some iterate of T is a contraction of X into itself. To each $x, y \in C[a, b]$ and $t \in [a, b]$, and, we have

$$|(T(x))(t) - (T(y))(t)|$$
$$= \left|\int_a^t (F(x(s), s) - F(y(s), s))\,ds\right| \le \int_a^t | F(x(s), s) - F(y(s), s) |\,ds \tag{3.4}$$
$$\le L\int | x(s) - y(s) |\,ds = L(t - a)\, d(x, y)$$

But

$$|(T^2(x))(t) - (T^2(y))(t)|$$
$$= |(T(T(x)))(t) - (T(T(y)))(t)| \le \int_a^t |F((T(x))(s), s) - F((T(y))(s), s)|\,ds$$
$$\le L\int_a^t |(T(x))(s) - (T(y))(s)|\,ds \le L^2 d(x,y)\int_a^t (s - a)\,ds$$
$$= \frac{L^2 (t - a)^2 d(x,y)}{2}$$

By induction, it is easy to see that

$$| (T^N(x))(t) - (T^N(x))(t) \le \frac{L^N (t - a)^N d(x, y)}{N!}.$$

It then follows that

$$d(T^N(x), T^N(y)) = \sup_{a \le t \le b} | (T^N(x))(t) - T^N(y))(t) |$$
$$\le \frac{L^N d(x, y) \sup_{a \le t \le b} (t - a)^N}{N!}$$
$$= \frac{L^N (b - a)^N d(x, y)}{N!} \tag{3.5}$$

Claim: $\frac{[L(b - a)]^N}{N!} \to 0$ as $N \to \infty$.

Proof of the claim: Let A be the smallest positive integer such that $L(b - a) \le A$. Then

$$\frac{[L(b - a)]^N}{N!} \le \frac{A^N}{N!} \tag{3.6}$$

But for $N > A$,

$$\frac{A^N}{N!} = \frac{A^A A^{N-A}}{A!\,(A+1)\,(A+2)\cdots(A+(N-A-1))\,(A+(N-A))}$$

$$= \frac{A^A}{A!}\,\frac{A}{A+1}\,\frac{A}{A+2}\cdots\frac{A}{N-1}\,\frac{A}{N}$$

and since the product $\frac{A}{A+1}\frac{A}{A+2}\cdots\frac{A}{N-1} < 1$, we have

$$\frac{A^N}{N!} < \frac{A^{A+1}}{A!\,N} \qquad (3.7)$$

Let $\varepsilon > 0$ be arbitrary. Choose $N' = \frac{1}{\varepsilon}\left(\frac{A^{A+1}}{A!}+1\right)$. Then

$$\frac{A^{A+1}}{A!} = N'\varepsilon - 1 < N'\varepsilon \le N\varepsilon \quad \forall N \ge N'.$$

Thus

$$\frac{A^{A+1}}{A!\,N} < \varepsilon \quad \forall N \ge N'. \qquad (3.8)$$

From (3.6), (3.7) and (3.8),

we have $\frac{[L(b-a)]^N}{N!} < \varepsilon \quad \forall N \ge N'$. Thus

$$\frac{[L(b-a)]^N}{N!} \to 0 \text{ as } N \to \infty. \qquad (3.9)$$

Clearly there exists a positive integer M such that $\frac{[L(b-a)]^M}{M!} < 1$ and hence from (3.5), we have

$$d(T^M(x), T^M(y)) \le \frac{[L(b-a)]^M}{M!}\, d(x,y).$$

It proves that T^M is a contraction of X into itself. By Theorem 2.2, T has a unique fixed point, say $\phi(t)$ and which is the unique solution to the differential equation (3.3). ◊

In practice, finding whether there is such an L as in Theorem 3.1 is a major question. This can be seen from the following theorem and some examples.

Theorem 3.2: *Let $F : R \times [a,b] \to R$ be a function of two variables such that $F(x,t)$ is defined for all $x \in R$ and $t \in [a,b]$. Assume that F is continuous, that F is partially differentiable with respect to x and that $\partial F / \partial x$ is bounded throughout $R \times [a,b]$. Then the differential equation $dx/dt = F(x,t)$ subject to an initial condition of the type $x(\alpha) = \beta$ has a unique solution.*

Proof: Assume that $|\partial F / \partial x| \le L$ for all $x \in R$ and $t \in [a,b]$. Fix t and define a function G of x alone by $G(x) = F(x,t)$. Clearly is differentiable and by mean value theorem for in R

$$G(x) - G(y) = G'(z)\,(x-y)$$

for some $z \in (x,y)$. Hence

$$|F(x,t) - F(y,t)| = |G'(z)|\,|x-y| = |\partial F(z,t)/\partial x|\,|x-y| \le L|x-y|.$$

The proof is complete by Theorem 3.1. ◊

Example 3.1: Let $F(x,t) = t/(2^t + x^2)$ for $x \in R$ and $t \in [-10,10]$. Then the differential equation $dx/dt = F(x,t)$ with some initial condition $x(\alpha) = \beta$ has a unique solution on $[-10,10]$. For, let p be any positive real number and x any real number. Then

$$P^{-1} - [2x/(P^2 + x^2)] = (P-x)^2/[P(P^2 + x^2)] \ge 0,$$

$$P^{-1} + [2x/(P^2 + x^2)] = (P+x)^2/[P(P^2 + x^2)] \ge 0$$

which yield $|2x/(P^2 + x^2)| \le P^{-1}$ so that

$$|2x/(P^2 + x^2)^2| \le |2x/(P^2 + x^2)|\,P^{-2} \le P^{-3}. \qquad (3.10)$$

Using (3.10) and $\partial F(x,t)/\partial x = -2tx/(2^t + x^2)^2$, we have

$$|\partial F/\partial x| = |t|\,|2x/(2^t + x^2)^2|$$

$$\le |t|\,|2x/((2^{t/2})^2 + x^2)^2| \le |t|\,[2^{t/2}]^{-3} \le 10 \times 2^{15}.$$

Since the same conclusion can be drawn for any interval $[-n,n]$ the given differential equation has a unique solution $x(t)$ defined for $t \in R$.

Example 3.2: Let $[a, b]$ be an interval contained in $(0, \infty)$. Let $F(x,t) = 1/(t + e^x)$ for $x \in \boldsymbol{R}$ and $t \in [a, b]$. Then the differential equation $dx/dt = F(x, t)$ with $x(\alpha) = \beta$ has a unique solution on $[a, b]$. For,

$$|\partial F/\partial x| = e^x/(t + e^x)^2 \le 1/(t + e^x) \le a^{-1}.$$

References

Rubinstein, I. and L. Rubinstein 1998. *Partial differential equations in classical mathematical physics*. Cambridge University Press, United Kingdom. 675 pp.

Bailey, D.F. 1966. Some theorems on contraction mappings. *Journal of London Mathematical Society* **41**: 101-106.

Yosida, K. 1978. *Functional analysis*. Springer, Berlin.

Braun, M. 1993. *Differential equations and their applications*, Springer-Verlag, New York. 578 pp.