Applications of Multiparameter Eigen value Problems

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ISSN : 2382-5359(Online), 1994-1412(Print)

DOI:
https://doi.org/10.3126/njst.v20i1.39390

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CONFILICT OF INTEREST: None

ABSTRACT

It was mainly due to Atkinson works, who introduced Linear Multiparameter Eigenvalue problems (LMEPs), based on determinantal operators on the Tensor Product Space. Later, in the area of Multiparameter eigenvalue problems has received attention from the Mathematicians in the recent years also, who pointed out that there exist a variety of mixed eigenvalue problems with several parameters in different scientific domains. This article aims to bring into a light variety of scientific problems that appear naturally as LMEPs. Of course, with all certainty, the list of collection of applications presented here are far from complete, and there are bound to be many more applications of which we are currently unaware. The paper may provide a review on applications of Multiparameter eigenvalue problems in different scientific domains and future possible applicatios both in theoretical and applied disciplines.

Keywords: Kronecker product, spectral theory, sturm-liouville eigenvalue problems, tensor product space.

1. INTRODUCTION

Linear Multiparameter parameter Eigenvalue Problems ($\mathbb{LMEP}$s) considered here is

$$\mathbb{W}_i(\lambda)x_1 := \left(Q_i - \sum_{j=1}^{k} \lambda_j P_{ij}\right)x_i = 0 \qquad (1.1)$$

where $\lambda_j \in \mathbb{C}$; $x_i \in \mathbb{C}^{n_i}$; and Q_i, P_{ij} are $n_i \times n_i$ over $\mathbb{C}$; $i, j := 1{:}k$. The problem (1.1) is extensively addressed in the thesis Bora (2019), where the problem is to find the k-tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{C}^k$ such that equation (1.1) has a solution $x_i \neq 0$ for $i := 1{:}k$, then such a λ is called eigenvalue and the corresponding tensor product $x := x_1 \otimes x_2 \otimes x_3 \otimes \dots \otimes x_k$ is called the eigenvector (right), where $\otimes$ stands for usual Kronecker product. Similarly, a tensor product $v = v_1 \otimes, \dots, \otimes v_k$ is called a left eigenvector if $v_i \neq 0$ and $v_i^* W_i(\lambda) = 0$ for $i := 1{:}k$. The history of the origin of the problem can be found in the domain of mathematical physics and are addressed in (Volkmer 1988, Cottin 2001). The spectral theory and its related classical results can be found in the works Atkinson (1968), Atkinson (1972) and Sleeman (1978) and in the papers (Hochstenbach 2003, Kosir 1994). Numerical solutions are analysed in Dong *et al.* (2016), Hochstenbach *et al.* (2002), Hochstenbach *et al.* (2008), Rodriguez (1969) and Xi (1996), and the references therein. In the study of the spectrum of 𝕃𝕄𝔼ℙ, the following commuting k-tuple of operators matrices is usually considered by the authors.

$$K_0 = \begin{vmatrix} P_{11} & P_{12} & \cdots & \cdots & P_{1k} \\ P_{21} & P_{22} & \cdots & \cdots & P_{2k} \\ \vdots & \vdots & \ddots & & \vdots \\ P_{k1} & P_{k2} & \cdots & \cdots & P_{kk} \end{vmatrix}_{\otimes} \qquad (1.2)$$

$$K_i := \begin{vmatrix} P_{11} & \cdots & P_{1,i-1} & Q_1 & P_{1,i+1} & \cdots & P_{1k} \\ P_{21} & \cdots & P_{2,i-1} & Q_2 & P_{2,i+1} & \cdots & P_{2k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ P_{k1} & \cdots & P_{k,i-1} & Q_k & P_{k,i+1} & \cdots & P_{kk} \end{vmatrix}_{\otimes} \qquad (1.3)$$

𝕃𝕄𝔼ℙs can be subdivided into two different categories, based on the different positivity conditions of the matrix operators K_0, defined in (1.2).

Definition 1.1 Kosir (1994): A 𝕃𝕄𝔼ℙ is called Hermitian, if all the matrices $\mathrm{P_{ij}}, \mathrm{i}, \mathrm{j} := 1 : \mathrm{k}$ defined in (1.1) are Hermitian, i.e. $\mathrm{B_{ij}} = \mathrm{B_{ij}^*}$

Definition 1.2 Hochstenbach et al. (2003): A 𝕃𝕄𝔼ℙ is called nonsingular, if the corresponding operator determinant $\mathrm{K_0}$ defined in (1.2) is nonsingular.

Definition 1.3 Hochstenbach et. al. (2002): A Hermitian 𝕃𝕄𝔼ℙ is called Right definite if

$$det \begin{pmatrix} x_1^* P_{11} x_1 & x_1^* P_{12} x_1 & \cdots & x_1^* P_{1k} x_1 \\ \vdots & \vdots & \ddots & \vdots \\ x_k^* P_{k1} x_k & x_k^* P_{k2} x_k & \cdots & x_k^* P_{kk} x_k \end{pmatrix} \geq \alpha \qquad (1.4)$$

for some $\alpha > 0$ and for all $x_i \in \mathbb{C}^{n_i}, \|x_i\| = 1, i := 1 : k$

Atkinson proved that Right definiteness is equivalent to the condition that the determinantal operator $K_0$is positive definite (Atkinson 1972). Existence of solutions of 𝕃𝕄𝔼ℙis assured for the right definite problem. This problem has been extensively studied over the years for the right definite and nonsingular case. Generally, for spectral analysis, the problem is considered as nonsingular. A nonsingular system (1.1) can be transformed into a system of joint generalised eigenvalue problems (GEPs) Atkinson (1972) of the form

$$K_i x = \lambda K_0 x \quad (1.5)$$

For nonsingular 𝕃𝕄𝔼ℙ the matrices $\Gamma_i := K_0^{-1} K_i$, $i := 1 : k$ commute. In this case, all eigenvalues of (1.1) agree with eigenvalues of (1.5).

2. SOME APPLICATIONS OF MULTIPARAMETER EIGENVALUE PROBLEMS

The multiparameter spectral theory finds its application in diverse scientific and engineering domains, particularly in some boundary-value problems, and in the problems of applied mathematics and functional analysis. The motivation for the numerical study of Multiparameter Eigenvalue problems for matrices comes from the discretisation of Multiparameter Sturm-Liouville eigenvalue problems in ordinary differential equations (Faierman 1969). Extensive coverage of research works on Multiparameter spectral theory of differential operators may be found in Atkinson *et al.(*2011), Faierman (1974, 1991), where Faierman considered the system of following differential equations

$$\frac{d^2}{dx_i^2} y_i(x_i) + q_i(x_i) y_i(x_i) + \sum_{j=1}^{k} \lambda_j a_{ij} y_i(x_i) = 0, i := 1 : k \qquad (2.1)$$

where $q_i(x_i), a_{ij};\ i, j := 1 : k$ are continuous, real valued and differentiable on the interval $[a_i, b_i]$ of real axis. The system (2.1) subject to the common boundary conditions

$$y_i(a_i) cos\alpha_i - y_i^{'}(a_i) sin\alpha_i = 0, \quad 0 \leq \alpha_i < \pi$$

$$y_i(b_i) cos\beta_i - y_i^{'}(b_i) sin\beta_i = 0, \quad 0 \leq \beta_i < \pi \qquad (2.2)$$

is the k-parameter of Sturm-Liouville system. We may formulate an eigenvalue problem for (2.1) by writing λ for $(\lambda_1, \lambda_2, \ldots, \lambda_k)$, where the problem is to choose λ in such that the equations (2.1) have non-trivial solutions satisfying the boundary conditions (2.2). More details on the system (2.1) are found in (Atkinson et al., 2011). If λ can be so chosen, then such a λ is called an eigenvalue and the corresponding product $\prod_{i=1}^{k} y_i(x_i)$ is called the eigenfunction. By discretisation techniques, e.g., the finite difference techniques Dai (2007), the Multiparameter Sturm-Liouville eigenvalue problems in terms of differential operators (2.1) can be converted into problems (1.1) in matrix form. 𝕃𝕄𝔼ℙs also arise in the theory of approximations, various body diffraction theory, and non-linear control problems. For the sake of completeness, some of the scientific problems which lead 𝕃𝕄𝔼ℙs are listed below:

2.1 Helmholtz Equation

Separation of variables applied to the Helmholtz equation of the form $\nabla^2 v + \omega^2 v = 0$

lead to 𝕃𝕄𝔼ℙ (Hochstenbach et al., 2019). They are concerned with elliptic, sphero-conal, parabolic, ellipsoidal, and prolate spheroidal coordinates (Plestanjak *et al*. 2015). A brief overview of these coordinate systems and related boundary value problems that yields 𝕃𝕄𝔼ℙ is presented below.

2.1.1 *Mathieus System*

Separation of variables applied to the two dimensional Helmholtz equation (2.3) in elliptic coordinates

$$x := cosh(\xi)cos(\eta);\ \ y := sinh(\xi)sin(\eta);\ \ 0 \leq \xi < \infty, 0 \leq \eta < 2\pi$$

yields $v(x,y) := G(\xi)F(\eta)$, where G and F satisfy a respective coupled system of Mathieu's angular and radial equations (Volkmer, 1988) as follows:

$G''(\eta) + (\lambda - 2\mu cos(2\eta))G(\eta) = 0(2.4)F''(\xi) + (\lambda - 2\mu cosh(2\eta))F(\xi) = 0$ (2.5)

where λ is the constant of separation, $\mu := \frac{1}{4}h^2w^2$, $h := \sqrt{\alpha^2 - \beta^2}$ with $\alpha := hcosh(\xi_0)$ (major axis) and $\beta := hsinh(\eta_0)$the minor axis of the membrane). These coupled systems Gheorghiu et al. (2012) of boundary value problems come from the problem of a vibrating elliptic membrane Ω with fixed boundaries $\partial\Omega$,

$$(\nabla^2 + \omega^2)v(x,y) = 0, (x,y) \in \Omega, v(x,y) = 0, (x,y) \in \partial\Omega$$

This problem, along with appropriate boundary conditions, is considered one of the most well-known examples of two-parameter eigenvalue problems Gheorghiu *et al*. (2012) and can be solved numerically using the Chebyshev collocation.

2.1.2 *Lame′S System*

Separation of variables applied to three dimensional Helmholtz equation (2.3) in sphero-conal coordinates.

$$x := rsin(\phi)(1 - k'^2cos^2(\theta))^{\frac{1}{2}}, y := rcos(\phi)(1 - k^2cos^2(\phi))^{\frac{1}{2}}, z: = rsin(\theta)sin(\phi)$$

where $r \geq 0, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi, 0 \leq k, k' \leq 1$ and $k^2 + k'^2 = 1$, gives $v(x,y) := P(r)Q(\phi)R(\theta)$, where P, Q and R satisfy the following system of differential equations:

$$r^2P''(r) + 2rP'(r) + [w^2r^2 - \rho(\rho+1)]P(r) = 0 \quad (2.6)$$

$$(1 - k^2cos^2(\phi))Q''(\phi) + k^2sin(\phi)cos(\phi)Q'(\phi) + [k^2\rho(\rho + 1)sin^2(\phi) + \delta]Q(\phi) = 0$$

$$(1 - k'^2cos^2(\theta))R''(\theta) + k'^2sin(\theta)cos(\phi)R'(\theta) + [k'^2\rho(\rho + 1)sin^2(\theta) - \delta]R(\theta) = 0 \quad (2.8)$$

Where $\rho(\rho+1)$ and δ are constant of separation, a system of equations (2.7)-(2.8) represents a trigonometric form of Lame′s system of differential equations, which forms a two-parameter eigenvalue problem along with boundary conditions. Similarly, a system consisting of all three equations (2.6)-(2.8) form a three-parameter eigenvalue problem together with boundary conditions. Numerical solution of these systems is reported in Boersma (1991) and (Willatzen 2003).

2.1.3 *Bessel Wave Equations*

Three-dimensional Helmholtz equation (2.3) in parabolic rotational coordinates

$$x := \xi\eta cos(\phi), \quad y := \xi\eta sin(\phi), \quad z := \frac{1}{2}(\eta^2 - \xi^2)$$

where $0 \leq \xi, \eta < \infty$, $0 \leq \phi \leq 2\pi$ lead to the solution $v(x,y) := X(\phi)Y(\xi)Z(\eta)$, where X, Y and Z satisfy

$$X''(\phi) + k_3^2X(\phi) = 0 \quad (2.9)$$

$$\xi^2Y''(\xi) + \xi Y'(\xi) + (k_2\xi^2 + w^2\xi^4 - k_3^2)Y(\xi) = 0 \quad (2.10)$$

$$\eta^2Z''(\eta) + \eta Z'(\eta) - (k_2\eta^2 - w^2\eta^4 + k_3^2)Z(\xi) = 0 \quad (2.11)$$

where $k_i, i := 2:3$ are constant of separation. Equation (2.9) gives $X(\phi) := e^{ip\phi}$, where $p = \pm k_3$. The parameter p will be an integer if the conditions $X(0) = X(2\pi)$, $X'(0) = X'(2\pi)$ is imposed on (2.9). Two Bassel's equations (2.10) and (2.11) under the suitable boundary conditions gives a two-parameter eigenvalue problem (Willatzen *et al*. 2011).

2.1.4 *Ellipsoidal Wave Equations*

The three-dimensional Helmholtz equation (2.3) is separable in ellipsoidal coordinates (α, β, γ)which is found in Section 29.18(ii) of (Olver 2010):

$$x := ksn(\alpha,k)sn(\beta,k)sn(\gamma,k)$$

$$y := -\frac{k}{k'}cn(\alpha,k)cn(\beta,k)cn(\gamma,k)$$

$$z := \frac{i}{kk'}dn(\alpha,k)dn(\beta,k)dn(\gamma,k)$$

which is a natural choice of the region $\Omega := \left\{(x,y,z): (\frac{x}{x_0})^2 + (\frac{y}{y_0})^2 + (\frac{z}{z_0})^2 \leq 1\right\}$ Hochstenbach *et al.* (2009), where sn, dn, cn denotes jacobian of elliptic functions defined with respect to theta functions as follows (2.7)

$$sn(\alpha,k) = \frac{\theta(0,\tau)\,\theta_{11}(x,\tau)}{\theta_{10}(0,\tau)\,\theta_{01}(x,\tau)}$$

$$cn(\alpha,k) = \frac{\theta_{01}(0,\tau)\,\theta_{10}(x,\tau)}{\theta_{10}(0,\tau)\,\theta_{01}(x,\tau)}$$

$$cn(\alpha,k) = \frac{\theta_{01}(0,\tau)\,\theta(x,\tau)}{\theta(0,\tau)\,\theta_{01}(x,\tau)}$$

Theta fuinctions are given by the formula θ(x, τ))=$\sum_{-\infty}^{\infty} q^{n^2}\eta^n$ such that $q = \exp(\pi i\,\tau)$ and $\eta = \exp(2\pi i\,x)$. The Theta functions with elliptic modulus k is given by$k = \left(\frac{\theta_{10}\,(0,\tau)}{\theta(0,\tau)}\right)^2$ and $\alpha = \pi\,\theta(0,\tau)^2\,z$. The Jacobi functions are defined in terms of elliptic modulus k(T), so we need to invert to find T in terms of k. Similarly other functions can be defined.

The solution can be written as

$$v(x(\alpha,\beta,\gamma),y(\alpha,\beta,\gamma),z(\alpha,\beta,\gamma)) := v_1(\alpha)v_2(\beta)v_3(\gamma) \quad (2.12)$$

where $v_i, i := 1:3$ satisfy ellipsoidal wave equations, and it can be expressed in Jacobian form as follows

$$v_1''(\alpha) + [h - \nu(\nu+1)k^2 sn^2(\alpha,k) + k^2w^2 sn^4(\alpha,k)]v_1(\alpha) = 0 \quad (2.13)$$

$$v_2''(\beta) + [h - \nu(\nu+1)k^2 sn^2(\beta,k) + k^2w^2 sn^4(\beta,k)]v_2(\beta) = 0 \quad (2.14)$$

$$v_3''(\gamma) + [h - \nu(\nu+1)k^2 sn^2(\gamma,k) + k^2w^2 sn^4(\gamma,k)]v_3(\gamma) = 0 \quad (2.15)$$

Where h, ν, and k are real parameters. The system (2.13)-(2.15) together with suitable boundary conditions admits a three-parameter eigenvalue problem, where each of the equations contains all three parameters. Using Multiparameter approach with spectral collocation techniques, computation of eigenvalue presented by Plestanjak *et al.* (2015) is quite efficient than other techniques presented in Willatzen *et al.* (2005) and Levinita (1999).

2.1.5 *System Of Spheroidal Wave Functions*

The equation (2.3) in prolate spheroidal coordinates $(\xi,\eta,\phi)$is found in Section 30.13(iv) of (Olver, 2010)that admits the solution $v(\xi,\eta,\phi) := v_1(\xi)v_2(\eta)v_3(\phi)$, where $v_i, i := 1:3$satisfy

$$(1-\xi^2)v_1''(\xi) - 2\xi v_1'(\xi) + \left(\lambda^2 + \gamma^2(1-\xi^2) - \frac{\mu^2}{1-\xi^2}\right)v_1(\xi) = 0 \quad (2.16)$$

$$(1-\eta^2)v_2''(\eta) - 2\eta v_2'(\eta) + \left(\lambda^2 + \gamma^2(1-\eta^2) - \frac{\mu^2}{1-\eta^2}\right)v_2(\eta) = 0 \quad (2.17)$$

$$v_3'''(\phi) + \mu^2 v_3(\phi) = 0 \quad (2.18)$$

Here $\gamma^2 = k^2c^2 \geq 0$, a three-parameter eigenvalue problem where one equation contains just one parameter and the other two of the equations contain all three parameters. If periodicity conditions $v_3(0) = v_3(2\pi), v_3'(0) = v_3'(2\pi)$ are imposed on (2.18), then parameter μ will become an integer. For a fixed μ of order 1000 to 10000, the system (2.16)-(2.17) together with boundary conditions, is solved as a two-parameter eigenvalue problem numerically in (Amodio *et al.* 2014). When separation of variable technique is applied to the equation (2.3) in oblate spheroidal coordinates, then a similar system of spheroidal wave functions is also found in Section 30.14(iv) of (Olver 2010).

2.2 SYSTEM OF POLYNOMIAL BUNDLES

The system polynomial bundles of the form $Au - \lambda Bu - \lambda^2 Cu = 0$, reported in Roach *et al.* (1977) and Roach (1979) can be replaced by an equivalent system of two-parameter eigenvalue problems, where the matrix operators A, B, C are Hermitian and $\lambda \in \mathbb{C}$. The study of polynomial bundles under the framework of Multiparameter spectral theory is much more reliable and efficient than the theory developed by (Gohberg *et al.* 1969).

2.3 Dielectrometry Sensors

When calculating the electrical properties of a material from measurements or inter-digital dielectrometry sensors Browne (2008), the material's properties with two layers are the eigenvalues, obtained from the corresponding two-parameter matrix eigenvalue problem.

2.4 Power Flow Equations

$\mathbb{LMEP}$s play a vital role in electrical engineering to find solution techniques of Power flow equations reported in (Molzahn 2010). Let P_k, Q_k, V_k and δ_k represent net real power injection, the net reactive power injection, the voltage magnitude and the voltage angle associated with each bus k of the power system. Each bus k in the power system can be categorised into three class: load (PQ) bus, slack bus and voltage controlled (PV) bus. Usually, a single bus is chosen as the slack bus, which has a fixed value of V_k and δ_k. Again, P_k and Q_k are calculated to form the power flow equations. The remaining buses are specified as either PQ or PV buses. For PQ bus V_k and δ_k and for PV bus Q_k and δ_k are calculated using power flow equation. In the derivation of the power flow equations, each bus's voltages are usually decomposed into orthogonal d and q components.

$$V_{dk} = V_k cos(\delta_k) \quad (2.19)$$

$V_{qk} = V_k sin(\delta_k)$ (2.20)

Using the equation for complex power $P + jQ = VI^* = VY^*V^*$, power flow equations are developed. For an n bus power system, the equations for the bus i becomes,

$$P_i + jQ_i = V_{di} + jV_{qi} \sum_{k=1}^{n} (G_{ik} - jB_{ik})(V_{dk} - jV_{qk}) \quad (2.21)$$

Equating real and imaginary parts of (2.21) and including the voltage magnitude relationship gives the complete set of power flow equations as follow:

$$P_i = V_{di} \sum_{k=1}^{n} (G_{ik}V_{dk} - B_{ik}V_{qk}) + V_{qi} \sum_{k=1}^{n} (B_{ik}V_{dk} + G_{ik}V_{qk}) \quad (2.22)$$

$$Q_i = V_{di} \sum_{k=1}^{n} (-B_{ik}V_{dk} - G_{ik}V_{qk}) + V_{qi} \sum_{k=1}^{n} (G_{ik}V_{dk} - B_{ik}V_{qk}) \quad (2.23)$$

$$V_i^2 = V_{di}^2 + V_{qi}^2 \quad (2.24)$$

These equations can be reformulated as $\mathbb{LMEP}$, and this reformulation shows an application of Multiparameter spectral theory in the power system. $2(n-1)$ parametric eigenvalue problems arise in n bus systems, where both q and d orthogonal components of bus voltages can be composed of corresponding eigenvalue and eigenvectors from the formulation of $\mathbb{LMEP}$s. Again, there are possible applications to study additional insights into solutions of the matrix formulation of the power flow equations. With the help of standard eigensolvers, the determination of several solutions to $\mathbb{LMEP}$ is useful for finding the stopping criteria for the continuation of power flow. Moreover, conditions of existence and uniqueness of solutions The multiparameter system is useful for evaluating the point of voltage collapse and analysing power system models in heavily loaded situations.

2.5 Elastomechanical Systems

In linear elastomechanical systems, the analytical models are generally updated by model parameter estimation either with input-output measurements or modal test results. This modal structure is a spatially discretised model, for example, a finite element model or a model of multibody systems consisting of a sum of matrices is multiplied by a dimensionless adjustment parameter. Cottin (2001) showed that updating linear analytical models can be converted to an $\mathbb{LMEP}$s that require only a minimum set of test data to find the value of actual model parameters with a negligibly low risk of biased estimates.

The governing equations of time-invariant spatially discretised elastomechanical n-d of the model are given by $A_2(a_2)u''(t) + A_1(a_1)u'(t) + A_0(a_0)u(t) = Sp(t)$ (2.25)

Where, A_0, A_1 and A_2 are respectively symmetric and positive definite stiffness matrix, damping matrix and inertia matrix of order n, with S being $n \times s$ input matrix with such that $s \leq n$. p(t) and u(t) are respective input and response vector. The measurement equation

$$y(t) = Hu(t)(+measurementnoise)$$

where H is the output matrix of order $m \times n$ such that $m \leq n$. By decomposing the model matrices $A_i;\ i := 0:2$ to a sum of matrices $A_{ir};\ i := 1:R_i$ (2.25) can be parametrised as

$$A_i(a_i) := \sum_{r=1}^{R_i} a_i A_{ir}$$

where the a_{ir} is the dimensionless adjustment parameter with $a_{ir} = 1$ for the a priori model. If the stiffness matrix is parametrised according to (2.25), provided the inertia and damping matrix are known, we obtain the following $\mathbb{LMEP}$ the undamped model.

$$(-\widehat{w}^2 A_2 + \sum_{s=1}^{k} a_{0s}A_{0s})x_i = 0, i := 1:k$$

where $\widehat{w} = 2\pi\hat{f}_i$ with natural frequencies of the system f_i, where $(\hat{.})$ denotes quantities gained by experiments.

2.6 Young-Frankel Scheme

In Young-Frankel scheme reported in Browne (2008), for the class separable partial differential equations of elliptic type in two independent variables, the eigenvalue of maximum modulus of certain two-parameter eigenvalue problem gives the optimum value of the over-relaxation parameter.

2.7 Aeroelastic Flutter Problems

Solution methods of Multiparameter eigenvalue problems can be used for the stability analysis of aeroelastic structures of flutter problems (Pons 2015). Let us consider a linear system with eigenvectors x, which depends arbitrarily with eigenvalue χ and another structural parameter $p \in \mathbb{R}$ such that $A(\chi, p)x = 0$

where $A \in \mathbb{C}^{n \times n}$.Taking complex conjugate of (2.29) and adding another equation $\bar{A}(\chi, p)\bar{x} = 0$

to the system, we get a Multiparameter eigenvalue problem. Consider a section model without damping, then governing equations of the model are

$$\begin{aligned} m\ddot{h} + d_h\dot{h} + k_h h - mx_\theta\ddot{\theta} &= -L(t) \\ I_P\ddot{\theta} + d_\theta\dot{\theta} + k_\theta\theta - mx_\theta\ddot{h} &= M(t) \end{aligned} \quad (2.31)$$

where m and I_p denote section mass and polar moment of inertia; k_h and k_θ denote section bending and twist stiffness; L(t) and M(t) denotes aerodynamic lift and moment and x_θ denotes section static imbalance. Taking Fourier to transform $[h(t),\theta(t)] = [\hat{h},\hat{\theta}]e^{i\chi t}$ of this section model we have

$$\begin{aligned}(-m\chi^2 + ld_h\chi + k_h)\hat{h} - mx_\theta\chi^2\hat{\theta} = L(\chi,\hat{h},\hat{\theta})\\ mx_\theta\chi^2\hat{h} + (-I_p\chi^2 + ld_\theta\chi + k_\theta)\hat{\theta} = M(\chi,\hat{h},\hat{\theta})\end{aligned} \quad (2.32)$$

To model the aerodynamic loads in the frequency domain,

$$\begin{aligned}L &= -\chi^2(L_h\hat{h} + L_\theta\hat{\theta})\\ M &= -\chi^2(M_h\hat{h} + M_\theta\hat{\theta})\end{aligned} \quad (2.33)$$

The aerodynamic coefficients $\{L_h, \mathrm{L}_\theta, \mathrm{M}_\mathrm{h}, \mathrm{M}_\theta\}$are a complex function of k. The final flutter problem takes the form $\left(\left(M_0 + G_0 + G_1\frac{1}{k} + G_2\frac{1}{k^2}\right)\chi^2 - D_0\chi - K_0\right)x = 0$(2.34)

with dimensionless parameter defined in Table 1.

$$\begin{aligned}G_0 &= \frac{1}{\mu}\begin{pmatrix}1 & a\\ a & (\frac{1}{8}+a^2)\end{pmatrix} G_1 = \frac{1}{\mu}\begin{pmatrix}-2t & 2t(1-a)\\ -t(1+2a) & ta(1-2a)\end{pmatrix}\\ G_2 &= \frac{1}{\mu}\begin{pmatrix}0 & 0\\ 2 & 1+2a\end{pmatrix} M_0 = \begin{pmatrix}1 & -r_\theta\\ -r_\theta & r^2\end{pmatrix}\\ D_0 &= \begin{pmatrix}2l\varsigma_h\omega_h & 0\\ 0 & 2lr^2\varsigma_\theta\omega_\theta\end{pmatrix} K_0 = \begin{pmatrix}\omega_h^2 & 0\\ 0 & r^2\omega_\theta\end{pmatrix}\end{aligned} \quad (2.35)$$

In γ -χ form it becomes

$$((M_0 + G_0)\chi^2 + G_1\gamma\chi + G_2\gamma^2 - D_0\chi - K_0)x = 0 \quad (2.36)$$

In $\tau - \lambda$ form it becomes

$$\left((M_0 + G_0) + G_1\tau + G_2\tau^2 - D_0\lambda - K_0\lambda^2\right)x = 0 \quad (2.37)$$

Table 1: Value of dimensionless parameter

Parameters	Values
Mass ratio - μ	$+20$
The radius of gyration - r	$+0.4899$
Bending damping - ς_h	$+1.4105$
Torsional damping - ς_θ	$+2.3508$
Bending nat. frequency - w_h	$+0.5642$ rad/s
Torsional nat. frequency - w_θ	$+1.4105$ rad/s
Static imbalance - r_θ	-0.1
Pivot point location - a	-0.2

For undamped system $D_0 = 0$

$$\left((M_0 + G_0) + G_1\tau + G_2\tau^2 - K_0\lambda\right)x = 0 \quad (2.38)$$

where $\Lambda = \lambda^2$.

$$\begin{aligned}\left((M_0 + G_0) + G_1\tau + G_2\tau^2 - K_0\Lambda\right)x = 0\\ \left((\bar{M}_0 + \bar{G}_0) + \bar{G}_1\tau + \bar{G}_2\tau^2 - \bar{K}_0\Lambda\right)\bar{x} = 0\end{aligned} \quad (2.39)$$

which are all quadratic polynomial eigenvalue problem. Using linearization techniques, this problem can be converted to a linear two-parameter eigenvalue problem.

2.8 Charge Singularity Problems

The governing equations of Charge singularity problem which are found in Morrison and Lewis (1976) and Bailey (1981) at the corner of a flat plate in the self-adjoint form are given by

$$((1-k^2cos^2x)^{\frac{1}{2}}L')' + (\lambda_1 + \lambda_2k^2sin^2x)(1-k^2cos^2x)^{-\frac{1}{2}}L = 0 \ on \ (0,\pi)$$

$$((1-k'^2cos^2y)^{\frac{1}{2}}N')' + (-\lambda_1 + \lambda_2k'^2sin^2y)(1-k'^2cos^2y)^{-\frac{1}{2}}n = 0 \ on \ (0,\pi) \quad (2.40)$$

subject to the boundary conditions

$L(0) = L'(\pi) = 0$ (2.41)

$N'(0) = N'(\pi) = 0 \ if \ 0 < x < \pi$ (2.42)

$N(0) = N(\pi) = 0 \ if \pi < x < 2\pi$ (2.43)

where $k{:}= sin(\frac{1}{2}|\pi - x|)$ and $k'{:}= cos(\frac{1}{2}|\pi - x|)$ and x is the angle of the sector. Using central difference techniques and by adopting Marcuk's identity Babuska *et. al.*(1966) with the grid $h = \frac{\pi}{n}$ and transforming boundary conditions equation (2.40) can be discretised to

$$(Q_1 + \lambda_1P_{11} + \lambda_2P_{12})\tilde{L} = 0; \ (Q_2 + \lambda_1P_{21} + \lambda_2P_{22})\tilde{N} = 0 \quad (2.44)$$

where Q_i, P_{ij} are $n \times n$ matrices over $\mathbb{R}$ for $i{:}= 1{:}2$, $\tilde{L}{:}= (\tilde{L}_1, \tilde{L}_2, \dots, \tilde{L}_n)^T$ with $\tilde{L}_i = L(ih)$, $i{:}= 1{:}n$. If $0 < x < \pi, Q_i, P_{ij} \in \mathbb{R}^{(n+1)\times(n+1)}, i{:}= 1{:}2$, $\tilde{N}{:}= (\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_n)^T$ with

$\tilde{N}_i = N(ih), i \coloneqq 1{:}n$. Similarly, if $\pi < x < 2\pi, Q_i, P_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$, $i{:}= 1{:}2$.

$\tilde{N}{:}= (\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_n)^T$ with $\tilde{N}_i = N(ih), i{:}= 1{:}(n-1)$. Equation (2.44) is a two parameter eigenvalue problem.

2.9 Bivariate Matrix Polynomials

Two-parameter bivariate matrix polynomials of degree n presented in the papers Plestanjak (2017) and Plestanjak et al. (2016) and are given by the equations (2.45) and (2.46)

$$\mathbb{M}_1(\lambda_1,\lambda_2){:}= \sum_{i=0}^{k}\sum_{j=0}^{k-i}\lambda_1^i\lambda_2^jV_{ij}x_1 = 0 \quad (2.45)$$

$$\mathbb{M}_2(\lambda_1,\lambda_2){:}= \sum_{i=0}^{k}\sum_{j=0}^{k-i}\lambda_1^i\lambda_2^jW_{ij}x_2 = 0 \quad (2.46)$$

where V_{ij}, W_{ij} are $n \times n$ matrices can be linearised ([24], Section 6) as an equivalent singular two-parameter eigenvalue problem with matrices of size $\frac{1}{2}k(k+1)n \times \frac{1}{2}k(k+1)n$.

This equivalent two-parameter eigenvalue problem helps the numeric of finding zeros of a system of bivariate matrix polynomials.

3. CONCLUSION

Multiparameter eigenvalue problems originated from applying the method of separation of variables techniques to solve partial differential equations of disparate scientific domains, especially in physics and engineering. Therefore, it has been concentrated on applications to boundary-value or eigenvalue problems for ordinary differential equations, particularly, the Multiparameter Sturm-Liouville Problem. However, there is still more scope for the further study of Multiparameter problems. As far as an abstract theory is concerned, Atkinson has introduced the finite-dimensional case of matrices. However, it would be of considerable interest to study the Multiparameter problems for difference operators also. This has enormous application in mathematical physics.The presented list of applications of the Multiparameter problems is not a complete one. There still exist possible applications of Multiparameter spectral theory both in theoretical and applied disciplines, and it will conduit new avenues for future research in this topic.

ACKNOWLEDGEMENT

The authors would like to acknowledge his gratitude to the associate managing editor and the three anonymous referees for their comments and suggestions to improve the paper. The author would also like to thank Dibrugarh University Institute of Engineering and Technology, Dibrugarh University, for providing all facility during research works.

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