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Dynamics on the Pre-periodic Components of the Fatou Set of Three Transcendental Entire Functions and Their Compositions

Bishnu Hari Subedi[1*] and *Ajaya Singh*[1]
[1]*Central Department of Mathematics, Institute of Science and Technology, Tribhuvan University, Nepal*

*Corresponding Author
subedi.abs@gmail.com, subedi_bh@cdmathtu.edu.np

Abstract
We prove that there exist three transcendental entire functions that can have an infinite number of domains which lie in the pre-periodic component of the Fatou set each of these functions and their compositions.

Keywords
Fatou set, carleman set, periodic component, pre-periodic component

Introduction
We denote the *complex plane* by $\mathbb{C}$, *extended complex plane* by $\mathbb{C}_\infty$ and *set of integers* greater than zero by $\mathbb{N}$. We assume the function $f: \mathbb{C} \to \mathbb{C}$ is an *entire function* unless stated otherwise. For any $n \in \mathbb{N}$, f^n always denotes the nth *iterate* of f. If $f^n(z) = z$ for some smallest $n \in \mathbb{N}$, then we say that z is a *periodic point* of period n. In particular, if $f(z) = z$, then z is a *fixed point* of f. If $|(f^n(z))'| < 1$, where $'$ represents complex differentiation of f^n with respect to z, then z is called an *attracting periodic point* of the function f. A family $\mathcal{F} = \{f : f$ is meromorphic on some domain X of $\mathbb{C}_\infty\}$ forms *normal family* if every sequence $(f)_{i\in\mathbb{N}}$ of functions contains a subsequence which converges uniformly to a finite limit or converges to ∞ on every compact subset D of X.

The *Fatou set* of f, denoted by F(f), is the set of points $z \in \mathbb{C}$ such that sequence $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighbourhood of z. A maximally connected subset of the Fatou set F(f) is called a *Fatou component*. By definition, the Fatou set is open and may or may not be empty. Fatou set is non-empty for every entire function with attracting periodic points. The complement of the Fatou set is called *Julia set* and, is denoted by J(f). Classically, it is known that the sets F(f) and J(f) form a fundamental partition of the complex plane $\mathbb{C}$.

Let $U \subset F(f)$ (a Fatou component) such that $f^n(U)$ for some $n \in \mathbb{N}$, is contained in some component of F(f), which is usually denoted by U_n. A Fatou component U is called pre-periodic if there exist integers $n > m \geq 0$ such that $U_n = U_m$. In particular, if $U_n = U_0 = U$ (that is, $f^n(U) \subset U$) for some smallest positive integer $n \geq 1$, then U is called *periodic Fatou component* of period n and $\{U_0, U_1 \ldots, U_{n-1}\}$ is called the *periodic cycle*. A component of Fatou set F(f) which is not pre-periodic is called *wandering domain*.

Our particular interest of this paper is whether there are more than two transcendental entire functions that have a similarity between the dynamics of their compositions and the dynamics of each of these functions. Dynamics of two transcendental entire functions and their compositions were studied by Singh, (2003). He constructed some examples of transcendental entire functions where dynamics of individual functions vary primarily from the dynamics of their compositions. Also, he constructed some examples of transcendental entire functions where the dynamics of individual functions is similar to the dynamics of their compositions. Dinesh Kumar, et al. (2015) extended the results of Singh, (2003) in a certain sense, to the possibility of having two transcendental entire functions that can have infinitely many domains which may lie in the pre-periodic component of the Fatou set of each function and their compositions. We (2019) investigated three transcendental entire functions such that there are infinitely many domains which lie in the wandering components of the Fatou set of each function and their compositions. In this paper, we investigate three transcendental entire functions such that each of individual functions as well as their compositions consists of an infinite number of domains which lie in the pre-periodic component of each of functions and their compositions. In particular, we prove the following result.

Theorem 1: *There exist three different transcendental entire functions* f, g *and* h, *and infinitelymany domains which lie in the different pre-periodic component of* F(f), F(g), F(h), F(f∘g), F(g∘f), F(f∘h), F(g∘h), F(h∘f), F(h∘g), F(f∘g∘h), F(f∘h∘g), F(g∘f∘h), F(g∘h ∘f), F(h∘f∘g) *and* F(h∘g∘f).

Carleman Set

To work out a Theorem 1, we need a notion in approximation theory of entire functions. In our case, we can use the notion of Carleman set from which we obtain an approximation of any holomorphic function by transcendental entire functions.

Definition 1 (Carleman set): *Let* F *be a closed subset of* ℂ *and* C(F)= {f:F→ ℂ:f *is continuous on* F *and analytic in the interior* F°*of* F}. *Then* F *is called a Carleman set* (for ℂ) *if for any* g∈C(F) *and any positive continuous function* ϵ *on* F, *there exists entire function* h *such that* |g(*z*)-h(*z*)|<ϵ *for all* *z*∈F.

Nersesjan proved the following important characterization of Carleman set in 1971, but we cite this from the work of Gaier (1987).

Proposition 1: *Let* F *be a proper subset of* ℂ. *Then* F *is a Carleman set for* ℂ *if and only if* F *satisfies the following conditions*:

a. $\mathbb{C}_\infty$ - F *is connected*;
b. $\mathbb{C}_\infty$ - F *is locally connected at* ∞;
c. *For every compact subset* K *of* ℂ, *there is a neighborhood V of* ∞ *in* $\mathbb{C}_\infty$ *such that no component of* F° *intersects both K and V.*

Note that the space $\mathbb{C}_\infty$ - F is connected if and only if each component Z of open set ℂ - F is unbounded. 1, This fact, together with Proposition1can be an excellent tool whether a set is a Carleman set for ℂ. The sets given in the following examples are Carleman sets for ℂ.

Example 1 (Gaier, Page 133 (1987)): The set E = {z∈ ℂ: |z| = 1, Re z > 0} ∪ {z =x: x> 1} $\cup\bigcup_{n=3}^{\infty}\{z = re^{i\theta}: r > 1, \theta = \pi/n\}$ is a Carleman set for ℂ.

Example 2 (Singh, Page 131 (2003)): Let E $=G_0\cup\bigcup_{n=1}^{\infty}(G_k \cup B_k \cup L_k \cup M_k)$, where

G_0 = {z∈ ℂ:|z-2| ≤ 1}, G_k = {z∈ℂ: |z - (4k +2)| ≤ 1} ∪{z ∈ℂ : Re z = 4k +2, Imz ≥ 1}∪{z ∈ℂ : Re z = 4k +2, Imz ≤ -1}, (k =1,2,3,…),B_k = {z ∈ℂ: |z + (4k +2)| ≤ 1} ∪{z ∈ ℂ : Re z = -(4k +2), Im z≥ 1} ∪ {z ∈ℂ : Re z = -(4k +2), Im z ≤ -1}, (k =1,2,3,…), L_k = {z ∈ ℂ: Re z = 4k},(k =1, 2, …) and M_k = {z ∈ ℂ: Re z = -4k}, (k =1, 2, 3…).

Then, by Preposition 1, we can easily show that E is a Carleman set for ℂ.

Proof of the main result (Theorem 1)

From the help of the Carleman set of Example 2, Dinesh Kumar, et al., (2016) proved the following result.

Proposition 2: *There are transcendental entire functions* f *and* g *such that there exist an infinite number of domains which lie in the pre-periodic component of the* F(f), F(g), F(f∘g) *and* F(g∘f).

Our main result, that is, Theorem 1, is an extension of Proposition 2. We proceeds for the following long proof of Theorem 1.

Proof of Theorem 1

Let,

E = $G_0\cup\bigcup_{n=1}^{\infty}(G_k \cup B_k \cup L_k \cup M_k)$.

Then E a Carleman set for $\mathbb{C}$ by Example 2. By the continuity of an exponential function, for given $\epsilon > 0$, there exists $\delta > 0$, may depend on a given point w_0, such that

$|w - w_0| < \delta \Longrightarrow |e^w - e^{w_0}| < \epsilon$.

If $w_0 = \log t$, then $e^{w_0} = e^{\log t} = t$. Let us choose $\epsilon = ½$. Then there exist sufficiently small $\delta_k > 0$ and $\delta_{k'} > 0$, such that

$|w-(\pi i + \log(4k-2))| < \delta_k \Longrightarrow |e^w + (4k-2)| < 1/2$, (k = 2, 3, 4…)

and

$|w - \log(4k-2)| < \delta_{k'} \Longrightarrow |e^w - (4k-2)| < 1/2$, (k =1, 2, 3…)

In particular, let us choose sufficiently small $\delta_0 > 0$, $\delta_1 > 0$, $\delta_2 > 0$, $\delta_{1'} > 0$ and $\delta_{2'} > 0$ such that

$|w - \log 2| < \delta_0 \Longrightarrow |e^w - 2| < 1/2$.
$|w - (\pi i + \log 6)| < \delta_1 \Longrightarrow |e^w + 6| < 1/2$.
$|w - (\pi i + \log 10)| < \delta_2 \Longrightarrow |e^w + 10| < 1/2$.
$|w - \log 6| < \delta_{1'} \Longrightarrow |e^w - 6| < 1/2$.
$|w - \log 10| < \delta_{2'} \Longrightarrow |e^w - 10| < 1/2$.

Next, let us define the following functions:

$$\alpha(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k); \\ \pi i + \log 6, & \forall z \in G_k, k = 1, 2, 3, \ldots; \\ \pi i + \log 10, & \forall z \in B_1; \\ \pi i + (4k-2), & \forall z \in B_k, k = 2, 3, 4 \ldots; \end{cases}$$

$$\beta(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k); \\ \pi i + \log 6, & \forall z \in G_1; \\ \pi i + \log 10, & \forall z \in B_1; \\ \log(4k-2), & \forall z \in G_k, k = 2, 3, 4 \ldots; \\ \pi i + (4k-2), & \forall z \in B_k, k = 2, 3, 4, \ldots; \end{cases}$$

$$\gamma(z) = \begin{cases} \log 2, & \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k); \\ \log 6, & \forall z \in B_k, k = 1, 2, 3, \ldots; \\ \log 10, & \forall z \in G_1; \\ \log(4k-2), & \forall z \in G_k, k = 2, 3, 4 \ldots; \end{cases}$$

Let us define again the following positive functions:

$$\epsilon_1(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k); \\ \delta_1, & \forall z \in G_k, k = 1, 2, 3 \ldots, \\ \delta_2, & \forall z \in B_1; \\ \delta_k, & \forall z \in B_k, k = 2, 3, 4 \ldots; \end{cases}$$

$$\epsilon_2(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k); \\ \delta_1, & \forall z \in G_1; \\ \delta_2, & \forall z \in B_1; \\ \delta_k, & \forall z \in B_k, k = 2, 3, 4 \ldots; \\ \delta_{k'}, & \forall z \in G_k, k = 2, 3, 4 \ldots; \end{cases}$$

$$\epsilon_3(z) = \begin{cases} \delta_0, & \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k); \\ \delta_{1'}, & \forall z \in B_k, k = 1, 2, 3 \ldots; \\ \delta_2, & \forall z \in G_1: \\ \delta_{k'} & \forall z \in G_k, k = 2, 3, 4 \ldots: \end{cases}$$

Clearly, the functions α(z), β(z) and γ(z) are piece wise constant functions, so they are continuous on the set E and analytic in E°. Also, since E is a Carleman set, so there exist an entire functions $f_1(z)$, $g_1(z)$ and $h_1(z)$ such that

$\forall z \in E$, $|f_1(z) - \alpha(z)| \le \epsilon_1(z)$, $|g_1(z) - \beta(z)| \le \epsilon_2(z)$ and $|h_1(z) - \gamma(z)| \le \epsilon_3(z)$.

Consequently, we get transcendental entire functions

$f(z) = e^{f_1(z)}$, $g(z) = e^{g_1(z)}$ and $h(z) = e^{h_1(z)}$

which respectively satisfy the following:

$|f(z) - 2| < ½$, $\forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;
$|f(z) + 6| < 1/2$, $\forall z \in G_k$, k =1, 2, 3…;
$|f(z) + 10| < 1/2$, $\forall z \in B_1$;
$|f(z) + (4k-2)| < 1/2$, $\forall z \in B_k$, k =2, 3, 4…;

$|g(z) - 2| < ½$, $\forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;
$|g(z) + 6| < 1/2$, $\forall z \in G_1$;
$|g(z) + 10| < 1/2$, $\forall z \in B_1$;
$|g(z) + (4k-2)| < 1/2$, $\forall z \in B_k$, k =2, 3, 4…;
$|g(z) - (4k-2)| < 1/2$, $\forall z \in G_k$, k =2,3,4…;

$|h(z) - 2| < ½$, $\forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;
$|h(z) - 6| < 1/2$, $\forall z \in B_k$, k =1, 2, 3…;
$|h(z) - 10| < 1/2$, $\forall z \in G_1$;
$|h(z) - (4k-2)| < 1/2$, $\forall z \in G_k$, k =2, 3, 4…;

As we did just above, each of the functions f, g and h map the domain $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup$

M_k) into smaller disk |z -2|< 1/2 contained in G_0 and each of this function is a contraction mapping. So, $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$ contains fixed points z_0, z_1 and z_2(say) such that

$f^n(G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)) \to z_0$ as $n \to \infty$

$g^n(G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)) \to z_1$ as $n \to \infty$

$h^n(G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)) \to z_2$ as $n \to \infty$

Thefixed points are respectively attracting fixed points for each function f, g and h, so $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$ lies in attracting cycle, and hence $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$ is a subset of each of the Fatou set F(f), F(g) and F(h). In this case, $J(f) \neq \mathbb{C}$, $J(g) \neq \mathbb{C}$ and $J(h) \neq \mathbb{C}$ and so Julia set of each of the function f, g and h do not contain interior-point and hence Fatou set of each of these function contains all interior points. In such case, Fatou set of each of the function f, g and h contain Carleman set E.

Again, as defined above, function f maps each G_k into smaller disk contained in B_1, B_1 into smaller disk contained in B_2 and each Bk for k =2,3,4,… into the smaller disk contained in B_{k-1}. G_k and B_k are contained in the pre-periodic components of Fatou set F(f) of the function f. Also, as function g maps each of the domains G_k into the smaller disk contained in G_{k-1} (k =2,3,4,…), G_1 into smaller disk contained in B_1, B_1 into the smaller disks contained in B_2 and B_k for k =2,3,4, … into smaller disk contained in B_{k-1}. In fact, G_k and B_k are contained in the pre-periodic components of the Fatouset F(g). Likewise, domains $G_k$and B_k, (k =1, 2, 3…) are contained in the pre-periodic components under the function h. Also, note that as we defined above, we can see that domains B_1 and B_2 lie in the periodic component of both of the functions f and g. From the above rule, we can say that domains $G_1$and G_2 lie in the periodic component of the function h.

Next, we examine the dynamical behavior of compositions of the functions f, g and h. The composite of any two and all three of these functions satisfy the following:

Dynamical behavior of f∘ g

$|(f \circ g)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;

$|(f \circ g)(z) + 10| < 1/2, \forall z \in G_1$;

$|(f \circ g)(z) + 6| < 1/2, \forall z \in G_k$, k =2, 3, 4…;

$|(f \circ g)(z) + 6| < 1/2, \forall z \in B_1$;

$|(f \circ g)(z) + 10| < 1/2, \forall z \in B_2$;

$|(f \circ g)(z) + (4k-6)| < 1/2, \forall z \in B_k$, k =2,3,4…;

This composition rule shows that the domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$, G_k and B_k, (k =1, 2, 3,…) belong to F(f ∘g) and in fact, each G_k and B_k for each $k \in \mathbb{N}$ is contained in the pre-periodic components of F(f ∘g). In particular, $B_1$and B_2 is contained in the periodic component of period 1 under the function f∘g.

Dynamical behavior of g∘ f

$|(g \circ f)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$

$|(g \circ f)(z) + 10| < 1/2, \forall z \in G_k$, k =1, 2, 3, 4…;

$|(g \circ f)(z) + 6| < 1/2, \forall z \in B_1$;

$|(g \circ f)(z) + 10| < 1/2, \forall z \in B_2$;

$|(g \circ f)(z) + (4k-6)| < 1/2, \forall z \in B_k$, k =3, 4, 5…;

From this composition rule, we can say that the domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$ and B_k, (k =1, 2, 3,…) belong to F (g ∘f) and in fact, each G_k and B_k for each $k \in \mathbb{N}$ is contained in the pre-periodic domain of F(f ∘g).In particular, each $B_1$and B_2 is contained in the periodic component of period 1 under the function g∘f.

Dynamical behavior of f ∘ h

$|(f \circ h)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$

$|(f \circ h)(z) + 6| < 1/2, \forall z \in G_1$;

$|(f \circ h)(z) + 6| < 1/2, \forall z \in G_k$, k =2, 3, 4…;

$|(f \circ h)(z) + 6| < 1/2, \forall z \in B_k$, k =1,2, 3…;

As we defined in the above composition rule, the domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$ and B_k, (k =1, 2, 3…) belong to F (f ∘ h). In fact, each G_k and B_k for all k =1, 2, 3, … contained in the pre-periodic components of F(f ∘ h).

Dynamical behavior of h ∘ f

$|(h \circ f)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;

$|(h \circ f)(z) - 6| < 1/2, \forall z \in G_k, k = 1, 2, 3\ldots$;
$|(h \circ g)(z) - 6| < 1/2, \forall z \in B_k, k = 2, 3, 4\ldots$;
$|(h \circ g)(z) - 6| < 1/2, \forall z \in B_1$;

From this composition rule, we can say that the domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$, G_k and B_k, (k =1, 2, 3…) belong to F(h ∘ f). In fact, each G_k and B_k, for all k =1,2,3… is contained in the pre-periodic component of F(h ∘ f).

Dynamical behavior of g ∘ h

$|(g \circ h)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;
$|(g \circ h)(z) - 6| < 1/2, \forall z \in G_1$;
$|(g \circ h)(z) + 6| < 1/2, \forall z \in G_k, k = 2, 3, 4\ldots$;
$|(g \circ h)(z) + 6| < 1/2, \forall z \in B_k, k = 1, 2, 3\ldots$;

As we defined in the above composition rule, the domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$, G_k and B_k, (k =1, 2, 3…) belong to F(g ∘ h). In fact, each G_k and B_k, for all k =1,2,3,…is contained in the pre-periodic component of F(h ∘ f).

Dynamical behavior of h ∘ g

$|(h \circ g)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;
$|(h \circ g)(z) - 6| < 1/2, \forall z \in G_1$;
$|(h \circ g)(z) - 10| < 1/2, \forall z \in G_2$;
$|(h \circ g)(z) - (4k - 6)| < 1/2, \forall z \in B_k, k = 3, 4, 5,\ldots$;
$|(h \circ g)(z) - 6| < 1/2, \forall z \in B_k, k = 1, 2, 3, \ldots$;

From this composition, we can say that the domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$, G_k and B_k, (k =1, 2, 3…) belong to F(h ∘ g). In fact, each G_k and B_k, for all k =1,2,3… is contained in the pre-periodic component of F(h ∘ f). In particular, each G_1 and G_2 is periodic component of period 1 under the function h∘ f.

Dynamical behavior of f∘ g∘ h

$|(f \circ g \circ h)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;
$|(f \circ g \circ h)(z) + 10| < 1/2, \forall z \in B_k, k = 1, 2, 3\ldots$;
$|(f \circ g \circ h)(z) + 6| < 1/2, \forall z \in G_1$;
$|(f \circ g \circ h)(z) + 10| < 1/2, \forall z \in G_2$;
$|(f \circ g \circ h)(z) + 6| < 1/2, \forall z \in G_k, k = 3, 4, 5\ldots$;

The composition rule assigned above tells us those domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$ G_k and B_k, (k =1, 2, 3…) lie in F (f ∘ g∘ h). In fact, each G_k and B_k for all k =1, 2, 3 … is contained in the pre-periodic component of F(f ∘ g∘ h).

Dynamical behavior of f ∘ h∘ g

$|(f \circ h \circ g)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;
$|(f \circ h \circ g)(z) + 6| < 1/2, \forall z \in G_1$;
$|(f \circ h \circ g)(z) + 6| < 1/2, \forall z \in G_2$;
$|(f \circ h \circ g)(z) + 6| < 1/2, \forall z \in G_k, k = 3, 4, 5\ldots$;
$|(f \circ h \circ g)(z) + 6| < 1/2, \forall z \in B_2$;
$|(f \circ h \circ g)(z) + 6| < 1/2, \forall z \in B_k, k = 2, 3, 4\ldots$;

The composition rule assigned above tells us that domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$, G_k and B_k, (k =1, 2, 3,…) lie in F(f ∘ h ∘ g) and in fact, each G_k and B_k, (k =1,2,3,…) is contained in the pre-periodic component of F(f ∘ h ∘ g). In particular, domain B_1 is contained in the periodic component under the function f∘ h∘ g.

Dynamical behavior of g ∘ f∘ h

$|(g \circ f \circ h)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;
$|(g \circ f \circ h)(z) + 10| < 1/2, \forall z \in G_k, k = 1, 2, 3\ldots$;
$|(g \circ f \circ h)(z) + 10| < 1/2, \forall z \in B_1$;
$|(g \circ f \circ h)(z) + 10| < 1/2, \forall z \in B_k, k = 2, 3, 4\ldots$;

The composition rule assigned above tells us that domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$, G_k and B_k, (k =1,2,3,…) lie in F (g ∘ f ∘ h). In fact, each G_k and B_k, for all k =1, 2, 3, … is contained in the pre-periodic component of F(g ∘ f∘ h). In particular, domain B_2 is contained in the periodic component under the function g ∘ f∘ h.

Dynamical behavior of g ∘ h ∘ f

$|(g \circ h \circ f)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;
$|(g \circ h \circ f)(z) + 6| < 1/2, \forall z \in G_k, k = 1, 2, 3, \ldots$;
$|(g \circ h \circ f)(z) + 6| < 1/2, \forall z \in B_1$;
$|(g \circ h \circ f)(z) + 6| < 1/2, \forall z \in B_k, k = 2, 3, 4\ldots$;

The composition rule assigned above tells us that domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$, G_k and B_k, (k =1, 2, 3…) lie in F (g∘ h∘ f). In fact, each G_k and B_k, for all $k = 1,2,3, \ldots$ is contained in the pre-periodic component of $F(g \circ h \circ f)$. In particular, domain B_1 is contained in the periodic component under the function $g \circ f \circ h$.

Dynamical behavior of h ∘ f ∘ g

$|(h \circ f \circ g)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;

$|(h \circ f \circ g)(z) - 6| < 1/2, \forall z \in G_k, k = 2, 3, 4 \ldots$;

$|(h \circ f \circ g)(z) - 6| < 1/2, \forall z \in G_1$;

$|(h \circ f \circ g)(z) - 6| < 1/2, \forall z \in B_1$;

$|(h \circ f \circ g)(z) - 6| < 1/2, \forall z \in B_2$;

$|(h \circ f \circ g)(z) - 6| < 1/2, \forall z \in B_k, k = 3, 4, 5 \ldots$;

The composition assigned above tells us that domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$, G_k and B_k, ($k = 1, 2, 3 \ldots$) lie in $F(h \circ f \circ g)$. In fact, each G_k and B_k for all $k = 1, 2, 3, \ldots$ is contained in the pre-periodic component of $F(h \circ f \circ g)$. In particular, domain G_1 is contained in the periodic component under the function $h \circ f \circ g$.

Dynamical behavior of h ∘ g ∘ f

$|(h \circ g \circ f)(z) - 2| < 1/2, \forall z \in G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$;

$|(h \circ g \circ f)(z) - 6| < 1/2, \forall z \in G_k, k = 1, 2, 3, \ldots$;

$|(h \circ g \circ f)(z) - 6| < 1/2, \forall z \in B_1$;

$|(h \circ g \circ f)(z) - 6| < 1/2, \forall z \in B_k, k = 2, 3, 4, \ldots$;

The composition rule assigned above tells us that domains $G_0 \cup \bigcup_{n=1}^{\infty}(L_k \cup M_k)$, G_k and B_k, ($k = 1,2,3, \ldots$) lie in $F(h \circ g \circ f)$. In fact, each G_k and B_k, for all $k = 1,2,3, \ldots$ is contained in the pre-periodic component of $F(h \circ g \circ f)$. In particular, domain G_1 is contained in the periodic component under the function $h \circ f \circ g$.

From all of the above discussion, we found that the domains G_k and B_k for all $k = 1,2,3, \ldots$ contained in the pre-periodic domains of the functions f, g, h and their compositions.

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