Supplementary Material

Inverse Exponentiated Odd Lomax Exponential Distribution: Properties and Applications

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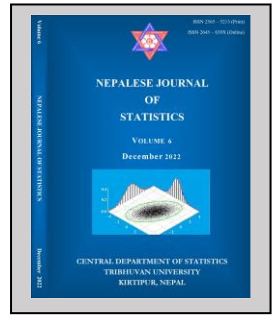

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SUPPLEMENTARY MATERIAL

(1) Formation of the Inverse Exponentiated Odd Lomax Exponential Distribution (IEOLE)

The cumulative distribution function of inverse exponential distribution is given as

$$G(x) = e^{-\alpha/x}, \qquad \overline{G}(x) = 1 - G(x) = 1 - e^{-\alpha/x}, x > 0, \alpha > 0$$

$$W(x) = \left[\frac{G(x)}{\overline{G}(x)}\right]^{\theta} = \left[\frac{e^{-\alpha/x}}{1-e^{-\alpha/x}}\right]^{\theta} = (e^{\alpha/x} - 1)^{-\theta}$$

$$F(x) = \int_0^{W(x)} r(t)dt$$

Here, r(t) is the generator which uses the probability density function(PDF) of the Lomax distribution(Chakrabortya, 2019) and hence, the PDF of the Lomax distribution as a generator is provided by

$$r(t) = \frac{\lambda}{\delta}\left\{1+\left(\frac{t}{\delta}\right)\right\}^{-(\lambda+1)} \;;\; t > 0, \lambda > 0, \delta > 0$$

(A) Derivation of CDF:

$$F(x,\alpha,\lambda,\theta,\delta) = \int_0^{(e^{\alpha/x}-1)^{-\theta}} \frac{\lambda}{\delta}\left\{1+\frac{t}{\delta}\right\}^{-(\lambda+1)} dt; \qquad = \frac{\lambda}{\delta}\int_0^{(e^{\alpha/x}-1)^{-\theta}} \left\{1+\frac{t}{\delta}\right\}^{-(\lambda+1)} dt;$$

Let $v = 1 + \frac{t}{\delta} \Rightarrow \frac{dv}{dt} = \frac{1}{\delta} \Rightarrow dt = \delta dv$

Now, $\int v^{-(\lambda+1)}\delta dv = \dfrac{\delta.v^{-(\lambda+1)+1}}{-(\lambda+1)+1} = \dfrac{\delta.v^{-\lambda}}{-\lambda} = \dfrac{\delta.(1+t/\delta)^{-\lambda}}{-\lambda}$

$$F(x,\alpha,\lambda,\theta,\delta) == \frac{\lambda}{\delta}\int_0^{(e^{\alpha/x}-1)^{-\theta}} \left\{1+\frac{t}{\delta}\right\}^{-(\lambda+1)} dt; = \frac{\lambda}{\delta}\left[\frac{\delta(1+t/\delta)^{-\lambda}}{-\lambda}\right]$$

$$= -\left[\left\{1+\frac{(e^{\alpha/x}-1)^{-\theta}}{\delta}\right\}^{-\lambda} -1\right] = 1-\left\{1+\frac{(e^{\alpha/x}-1)^{-\theta}}{\delta}\right\}^{-\lambda} = 1-\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-\lambda}$$

(2) **Derivation of Probability density function of the Inverse Exponentiated Odd Lomax Exponential Distribution (IEOLE)**

$$f(x) = \frac{dF(x)}{dx} = \frac{d\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-\lambda}\right]}{dx} = -\frac{d\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-\lambda}}{dx}$$

$$= \frac{\lambda}{\delta}\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-(\lambda+1)} \times(-\theta)(e^{\alpha/x}-1)^{-\theta-1}\frac{de^{\alpha/x}}{dx}$$

$$= \lambda\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-\lambda-1}\frac{1}{\delta}\frac{d(e^{\alpha/x}-1)^{-\theta}}{dx}$$

$$= -\frac{\lambda\theta}{\delta}\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-(\lambda+1)}(e^{\alpha/x}-1)^{-(\theta+1)}\alpha e^{\alpha/x}(-1/x^2)$$

$$= \frac{\lambda\theta\alpha}{\delta}x^{-2}e^{\alpha/x}\left(e^{\alpha/x}-1\right)^{-(\theta+1)}\left[1+\frac{1}{\delta}\left\{e^{\alpha/x}-1\right\}^{-\theta}\right]^{-(\lambda+1)}$$

(3) **Verification of Probability density function of the Inverse Exponentiated Odd Lomax Exponential Distribution (IEOLE)**

To verify $\int_0^\infty f(x)dx = 1$

Now, definite integral= $\displaystyle\int_0^\infty \frac{\lambda\theta\alpha}{\delta}x^{-2}e^{\alpha/x}\left(e^{\alpha/x}-1\right)^{-(\theta+1)}\left[1+\frac{1}{\delta}\left\{e^{\alpha/x}-1\right\}^{-\theta}\right]^{-(\lambda+1)} dx$

Let $1+\dfrac{1}{\delta}\left\{e^{\alpha/x}-1\right\}^{-\theta} = v \Rightarrow \dfrac{dv}{dx} = \dfrac{\alpha\theta}{\delta}\dfrac{e^{\alpha/x}}{x^2}(e^{\alpha/x}-1)^{-(\theta+1)} \Rightarrow dv = \dfrac{\alpha\theta}{\delta}\dfrac{e^{\alpha/x}}{x^2}(e^{\alpha/x}-1)^{-(\theta+1)}dx$

Now, indefinite integral= $\lambda\int[v]^{-(\lambda+1)}dv = \dfrac{\lambda(v)^{-(\lambda+1)+1}}{-(\lambda+1)+1} = \dfrac{\lambda(v)^{-\lambda}}{-\lambda} = -(v)^{-\lambda}$ =

$$-\left(1+\frac{1}{\delta}\left\{e^{\alpha/x}-1\right\}^{-\theta}\right)^{-\lambda}$$

∴ Definite integral = $\left[-\left(1+\dfrac{1}{\delta}\left\{e^{\alpha/x}-1\right\}^{-\theta}\right)^{-\lambda}\right]_0^\infty$

$$=\left[-\left(1+\frac{1}{\delta}\left\{e^{\alpha/x}-1\right\}^{-\theta}\right)^{-\lambda}\right]_0^{\infty}=-\left(1+\frac{1}{\delta}\left\{e^{\alpha/\infty}-1\right\}^{-\theta}\right)^{-\lambda}+\left(1+\frac{1}{\delta}\left\{e^{\alpha/0}-1\right\}^{-\theta}\right)^{-\lambda}$$

$$=-\left(1+\frac{1}{\delta}\left\{e^{0}-1\right\}^{-\theta}\right)^{-\lambda}+\left(1+\frac{1}{\delta}\left\{e^{\infty}-1\right\}^{-\theta}\right)^{-\lambda}$$

$$=-\left(1+\frac{1}{\delta}\left\{1-1\right\}^{-\theta}\right)^{-\lambda}+\left(1+\frac{1}{\delta}\left\{\infty-1\right\}^{-\theta}\right)^{-\lambda}$$

$$=-\left(1+\frac{1}{\delta(0)^{\theta}}\right)^{-\lambda}+\left(1+\frac{1}{\delta(\infty)^{\infty}}\right)^{-\lambda}=$$

$$-(\infty)^{-\lambda}+(1+0)^{-\lambda}=-\frac{1}{(\infty)^{\lambda}}+1=0+1=1 \text{ verified}$$

(4) Derivations of Survival function /Reliability function

$$R(x)=1-F(x)=1-\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-\lambda}\right]\quad=\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-\lambda}$$

(5) Derivation of Quantile function

The quantile function is

$$Q(u)=F^{-1}(u)$$

Let F(x) = u .Then, $x=F^{-1}(u)$ (1)

That is, $1-\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-\lambda}=u$, 0 <u<1.

or, $\left\{1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}\right\}^{-\lambda}=1-u$ or, $1+\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}=(1-u)^{-1/\lambda}$

or, $\frac{1}{\delta}(e^{\alpha/x}-1)^{-\theta}=(1-u)^{-1/\lambda}-1$, or, $(e^{\alpha/x}-1)^{-\theta}=\delta\left[(1-u)^{-1/\lambda}-1\right]$

or, $e^{\alpha/x}-1=\left[\delta\left\{(1-u)^{-1/\lambda}-1\right\}\right]^{-1/\theta}=e^{\alpha/x}=1+\left[\delta\left\{(1-u)^{-1/\lambda}-1\right\}\right]^{-1/\theta}$

or, $\frac{\alpha}{x}=\log\left[1+\left[\delta\left\{(1-u)^{-1/\lambda}-1\right\}\right]^{-1/\theta}\right]$

or, $x=\alpha\left[\log\left[1+\left[\delta\left\{(1-u)^{-1/\lambda}-1\right\}\right]^{-1/\theta}\right]\right]^{-1}$ (2)

Hence, $Q(u)=F^{-1}(u)=\alpha\left[\log\left[1+\left[\delta\left\{(1-u)^{-1/\lambda}-1\right\}\right]^{-1/\theta}\right]\right]^{-1}$ (3)

(6) Derivation of Median

For median, taking u= ½ in equation (3), we get

$$Median=\alpha\left[\log\left[1+\left[\delta\left\{(1/2)^{-1/\lambda}-1\right\}\right]^{-1/\theta}\right]\right]^{-1}$$

(7) **Derivation of Random deviate generation**

The expression (2) is the expression for random deviate generation which helps to generate random numbers (observations).

(8) **Derivation of log likelihood function and their partial derivatives with respect to four parameters**

$$l=n\ln\alpha+n\ln\lambda+n\ln\theta-n\ln\delta-2\sum_{i=1}^{n}\ln(x_i)-(\theta+1)\sum_{i=1}^{n}\ln(e^{\alpha/x_i}-1)+\alpha\sum_{i=1}^{n}(1/x_i)$$
$$-(\lambda+1)\sum_{i=1}^{n}\ln\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}$$

Differentiating partially with respect to α , we get

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}-(\theta+1)\sum_{i=1}^{n}\frac{d\ln(e^{\alpha/x_i}-1)}{d\alpha}+\sum_{i=1}^{n}(1/x_i)-(\lambda+1)\sum_{i=1}^{n}\frac{\partial\ln\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}}{\partial\alpha}$$

$$=\frac{n}{\alpha}-(\theta+1)\sum_{i=1}^{n}(e^{\alpha/x_i}-1)^{-1}\times\frac{1}{x_i}e^{\alpha/x_i}+\sum_{i=1}^{n}(1/x_i)$$
$$-(\lambda+1)\sum_{i=1}^{n}\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}^{-1}\left\{\frac{1}{\delta}(-\theta)(e^{\alpha/x_i}-1)^{-(\theta+1)}\times(\frac{1}{x_i})e^{\alpha/x_i}\right\}$$

$$=\frac{n}{\alpha}-(\theta+1)\sum_{i=1}^{n}\frac{1}{x_i}(e^{\alpha/x_i}-1)^{-1}e^{\alpha/x_i}+\sum_{i=1}^{n}(1/x_i)+\frac{(\lambda+1)\theta}{\delta}\sum_{i=1}^{n}\frac{e^{\alpha/x_i}}{x_i}\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}^{-1}(e^{\alpha/x_i}-1)^{-(\theta+1)}$$

Differentiating with respect to λ , we get

$$\frac{\partial l}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}\ln\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}$$

Differentiating with respect to θ , we get

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\ln(e^{\alpha/x_i}-1)-(\lambda+1)\sum_{i=1}^{n}\frac{\partial\ln\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}}{\partial\theta}$$

$$=\frac{n}{\theta}-\sum_{i=1}^{n}\ln(e^{\alpha/x_i}-1)-(\lambda+1)\sum_{i=1}^{n}\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}^{-1}\frac{1}{\delta}\frac{\partial(e^{\alpha/x_i}-1)^{-\theta}}{\partial\theta}$$

$$=\frac{n}{\theta}-\sum_{i=1}^{n}\ln(e^{\alpha/x_i}-1)-\frac{(\lambda+1)}{\delta}\sum_{i=1}^{n}\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}^{-1}\ln(e^{\alpha/x_i}-1).(e^{\alpha/x_i}-1)^{-\theta}.(-1)$$

$$=\frac{n}{\theta}-\sum_{i=1}^{n}\ln(e^{\alpha/x_i}-1)+\frac{(\lambda+1)}{\delta}\sum_{i=1}^{n}\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}^{-1}(e^{\alpha/x_i}-1)^{-\theta}\ln(e^{\alpha/x_i}-1).$$

Differentiating with respect to δ, we get

$$\frac{\partial l}{\partial \delta}=-\frac{n}{\delta}-(\lambda+1)\sum_{i=1}^{n}\frac{\partial \ln\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}}{\partial \delta}$$

$$=-\frac{n}{\delta}-(\lambda+1)\sum_{i=1}^{n}\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}^{-1}\left\{-\frac{1}{\delta^2}(e^{\alpha/x_i}-1)^{-\theta}\right\}$$

$$=-\frac{n}{\delta}+\frac{(\lambda+1)}{\delta^2}\sum_{i=1}^{n}(e^{\alpha/x_i}-1)^{-\theta}\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}^{-1}$$

(9) **Derivation of function A and their partial derivatives with respect to four parameters**

$$A(x;\alpha,\lambda,\theta,\delta)=\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^2$$

Or, $A\left(x_{(i)};\alpha,\lambda,\theta,\delta\right)=\sum_{i=1}^{n}\left[\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x(i)}-1)^{-\theta}\right\}^{-\lambda}\right]-\frac{i}{n+1}\right]^2$; $x>0,(\alpha,\lambda,\theta,\delta)>0$

Differentiating A with respect to α , we get

$$\frac{\partial A}{\partial \alpha}=\frac{\partial \sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^2}{\partial \alpha}$$

$$=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\frac{\partial}{\partial \alpha}\left[\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}\right]-\frac{i}{n+1}\right]$$

$$=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right][-(-\lambda)]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}\times\frac{1}{\delta}(-\theta)(e^{\alpha/x_{(i)}}-1)^{-(\theta+1)}\times(\frac{1}{x_{(i)}})e^{\alpha/x_{(i)}}$$

$$=\frac{-2\lambda\theta}{\delta}\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}\frac{e^{\alpha/x_{(i)}}}{x_{(i)}}(e^{\alpha/x_{(i)}}-1)^{-(\theta+1)}$$

Differentiating A with respect to λ , we get

$$\frac{\partial A}{\partial \lambda}=\frac{\partial \sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^2}{\partial \lambda}=\frac{\partial \sum_{i=1}^{n}\left[\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}\right]-\frac{i}{n+1}\right]^2}{\partial \lambda}$$

$$=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\frac{\partial}{\partial \lambda}\left[\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}\right]-\frac{i}{n+1}\right]$$

$$=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right][-(-1)]\ln[1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}$$

$$=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\left[\ln\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}$$

Differentiating A with respect to θ, we get

$$\frac{\partial A}{\partial \theta}=\frac{\partial \sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^{2}}{\partial \theta}=\frac{\partial \sum_{i=1}^{n}\left[\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}\right]-\frac{i}{n+1}\right]^{2}}{\partial \theta}$$

$$=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\frac{\partial}{\partial \theta}\left[\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}\right]-\frac{i}{n+1}\right]$$

$$=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\left[-(-\lambda)\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}\times\frac{\partial}{\partial \theta}\left\{\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}$$

$$=\frac{2\lambda}{\delta}\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}\left[(e^{\alpha/x_{(i)}}-1)^{-\theta}\ln(e^{\alpha/x_{(i)}}-1)\right](-1)$$

$$=-\frac{2\lambda}{\delta}\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}\left[(e^{\alpha/x_{(i)}}-1)^{-\theta}\ln(e^{\alpha/x_{(i)}}-1)\right]$$

Differentiating A with respect to δ, we get

$$\frac{\partial A}{\partial \delta}=\frac{\partial \sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^{2}}{\partial \delta}=\frac{\partial \sum_{i=1}^{n}\left[\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}\right]-\frac{i}{n+1}\right]^{2}}{\partial \delta}$$

$$=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\frac{\partial}{\partial \delta}\left[\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}\right]-\frac{i}{n+1}\right]$$

$$=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\left[(-1)(-\lambda)\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}\times(-\frac{1}{\delta^{2}})(e^{\alpha/x_{(i)}}-1)^{-\theta}$$

$$=-\frac{2\lambda}{\delta^{2}}\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}(e^{\alpha/x_{(i)}}-1)^{-\theta}$$

(I) Derivation of function Z and their partial derivatives with respect to four parameters

$$Z\left(X_{(i)};\alpha,\lambda,\theta,\delta\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(x_{i:n}\,|\,\alpha,\lambda,\theta,\delta\right)-\frac{2i-1}{2n}\right]^{2}$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left[1-\left\{1+\frac{1}{\delta}(e^{\alpha/x(i)}-1)^{-\theta}\right\}^{-\lambda}\right]-\frac{2i-1}{2n}\right]^{2}$$

Since $\frac{1}{12n}$ is constant having derivative zero, replacing 2/(n+1) in LSE by (2n-1)/2n, we can get the partial derivatives $\frac{\partial Z}{\partial \alpha}$, $\frac{\partial Z}{\partial \lambda}$, $\frac{\partial Z}{\partial \theta}$ and $\frac{\partial Z}{\partial \delta}$ as

$$\frac{\partial Z}{\partial \alpha}=\frac{-2\lambda\theta}{\delta}\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{2i-1}{2n}\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}\frac{e^{\alpha/x_{(i)}}}{x_{(i)}}(e^{\alpha/x_{(i)}}-1)^{-(\theta+1)}$$

$$\frac{\partial Z}{\partial \lambda}=2\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{2i-1}{n}\right]\left[\ln\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-\lambda}$$

$$\frac{\partial Z}{\partial \theta}=-\frac{2\lambda}{\delta}\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{2i-1}{2n}\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}\left[(e^{\alpha/x_{(i)}}-1)^{-\theta}\ln(e^{\alpha/x_{(i)}}-1)\right]$$

$$\frac{\partial Z}{\partial \delta}=-\frac{2\lambda}{\delta^{2}}\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{2i-1}{2n}\right]\left\{1+\frac{1}{\delta}(e^{\alpha/x_{(i)}}-1)^{-\theta}\right\}^{-(\lambda+1)}(e^{\alpha/x_{(i)}}-1)^{-\theta}$$

(2) Models considered for comparison

Generalized Rayleigh (GR) distribution:

$$f_{GR}(x)=2\alpha\lambda^{2}xe^{-(\lambda x)^{2}}\left\{1-e^{-(\lambda x)^{2}}\right\}^{\alpha-1}\quad;\quad x>0,(\alpha,\lambda)>0$$

Exponentiated Power Lindley (EPL) distribution:

$$f_{EPL}(x)=\frac{\alpha\beta\theta^{2}x^{\beta-1}}{(\theta+1)}\left(1+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\theta+1}\right)e^{-\theta x^{\beta}}\right]^{\alpha-1};x>0,,$$

Generalized Weibull Extension (GWE):

$$f_{GWE}(x)=\alpha\,\beta\left(\lambda x\right)^{\beta-1}\exp\left\{\left(\lambda x\right)^{\beta}+\frac{1}{\lambda}\left(1-\exp\left(\left(\lambda x\right)^{\beta}\right)\right)\right\}$$
$$\left[1-\exp\left\{\frac{1}{\lambda}\left(1-\exp\left(\left(\lambda x\right)^{\beta}\right)\right)\right\}\right]^{\alpha-1}\quad;\quad x\geq 0,\,(\alpha,\beta,\lambda)>0$$

Exponentiated Weibull Distribution (EW):

$$f_{EW}(x)=\alpha\beta\lambda x^{\beta-1}\exp\left(-\alpha x^{\beta}\right)\left(1-\exp\left(-\alpha x^{\beta}\right)\right)^{\lambda-1}\quad x\geq 0,\,(\alpha,\beta,\lambda)>0$$

Generalized Exponential (GE) distribution:

$$f_{GE}(x)=\alpha\lambda e^{-\lambda x}\left\{1-e^{-\lambda x}\right\}^{\alpha-1}\quad \alpha>0,\lambda>0,x\geq 0$$

Half Logistic Nadarajah Haghighi (HLNHE) Distribution:

$$f_{HLNHE}(x)=\frac{2\alpha\beta\lambda(1+\alpha x)^{(\beta-1)}\,exp\left(\lambda\left(1-(1+\alpha x)^{\beta}\right)\right)}{\left[1+exp\left(\lambda\left(1-(1+\alpha x)^{\beta}\right)\right)\right]^{2}}\quad;\alpha,\beta,\lambda>0,x>0$$

Lindley Inverse Weibull (LIW):

$$f_{LIW}(x) = \alpha\beta\left(\frac{\theta^2}{\theta+1}\right)x^{-(\beta+1)}\exp\left(-\alpha x^{-\beta}\right)\left(1-\exp\left(-\alpha x^{-\beta}\right)\right)^{\alpha-1}$$
$$\left\{1-\log\left(1-\exp\left(-\alpha x^{-\beta}\right)\right)\right\} \qquad : (\alpha,\beta,\theta) > 0, x > 0$$

Exponentiated Generalized inverted Exponential (EGIE):

$$f_{EGIE}(x) = \frac{\alpha\beta\lambda}{x^2}e^{-\lambda/x}\left\{1-e^{-\lambda/x}\right\}^{\alpha-1}\left[1-\left\{1-e^{-\lambda/x}\right\}^{\alpha}\right]^{\beta-1} ;\alpha,\beta,\lambda > 0, x > 0$$

Lomax Exponentiated Weibull (LEW):

$$f_{LEW}(x) = \alpha\beta\theta x^{\beta-1}\left(1-e^{-x^{\beta}}\right)^{\alpha-1}\left[1-\left(1-e^{-x^{\beta}}\right)^{\alpha}\right]^{2}$$
$$\left[1+\left(1-e^{-x^{\beta}}\right)^{\alpha}\left\{1-\left(1-e^{-x^{\beta}}\right)^{\alpha}\right\}^{-1}\right]^{-(\theta+1)} ;(\alpha,\beta,\theta) > 0, x > 0$$

Generalized Inverted Generalized Exponential (GIGE):

$$f_{GIGE}(x) = \alpha\lambda\gamma x^{-2}e^{-\gamma(\lambda/x)};\left(1-e^{-\gamma(\lambda/x)}\right)^{\alpha-1} ;\alpha,\lambda,\gamma > 0 , x > 0 ,$$

(10) **Detailed calculation for mode**

$$f(x) = \begin{cases} \dfrac{\lambda\theta\alpha}{\delta}x^{-2}e^{\alpha/x}\left(e^{\alpha/x}-1\right)^{-(\theta+1)}\left[1+\dfrac{1}{\delta}\left(e^{\alpha/x}-1\right)^{-\theta}\right]^{-(\lambda+1)} & ; x \ge 0, (\alpha,\lambda,\theta,\lambda) > 0 \\ 0 & ; \text{Otherwise} \end{cases}$$

log density function is

Differentiating with respect to x, we get

$$\ln f(x) = \ln\alpha + \ln\lambda + \ln\theta - \ln\delta - 2\ln x + \alpha x^{-1} - (\theta+1)\ln(e^{\alpha/x_i}-1) - (\lambda+1)\ln\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\} \qquad (15)$$

$$\frac{d\ln f(x)}{dx}=\frac{d}{dx}\begin{bmatrix}\ln f(x)=\ln\alpha+\ln\lambda+\ln\theta-\ln\delta-2\ln x+\\ \alpha x^{-1}-(\theta+1)\ln(e^{\alpha/x_i}-1)-(\lambda+1)\ln\left\{1+\frac{1}{\delta}(e^{\alpha/x_i}-1)^{-\theta}\right\}\end{bmatrix}$$

$$\frac{f^{'}(x)}{f(x)}=-\frac{2}{x}-\frac{\alpha}{x^2}+\frac{\alpha(\theta+1)e^{\alpha/x}}{x^2\left(e^{\alpha/x}-1\right)}-\frac{\alpha\theta(\lambda+1)e^{\alpha/x}\left(e^{\alpha/x}-1\right)^{-(\theta+1)}}{x^2\left\{\delta+\left(e^{\alpha/x}-1\right)^{-\theta}\right\}}$$

$$f^{'}(x)=\left[-\frac{2}{x}-\frac{\alpha}{x^2}+\frac{\alpha(\theta+1)e^{\alpha/x}}{x^2\left(e^{\alpha/x}-1\right)}-\frac{\alpha\theta(\lambda+1)e^{\alpha/x}\left(e^{\alpha/x}-1\right)^{-(\theta+1)}}{x^2\left\{\delta+\left(e^{\alpha/x}-1\right)^{-\theta}\right\}}\right]f(x)$$

applying $f^{'}(x)=0$

$$\left[-\frac{2}{x}-\frac{\alpha}{x^2}+\frac{\alpha(\theta+1)e^{\alpha/x}}{x^2\left(e^{\alpha/x}-1\right)}-\frac{\alpha\theta(\lambda+1)e^{\alpha/x}\left(e^{\alpha/x}-1\right)^{-(\theta+1)}}{x^2\left\{\delta+\left(e^{\alpha/x}-1\right)^{-\theta}\right\}}\right]=0,\text{Since } f(x)\neq 0$$

$$\frac{\alpha\theta(\lambda+1)e^{\alpha/x}\left(e^{\alpha/x}-1\right)^{-(\theta+1)}}{x\left\{\delta+\left(e^{\alpha/x}-1\right)^{-\theta}\right\}}-\frac{\alpha(\theta+1)e^{\alpha/x}}{x\left(e^{\alpha/x}-1\right)}+\frac{\alpha}{x}+2=0$$

Solving above equation, mode can be calculated.

Since this is non linear so it needs Newton-Raphson method for solution.