Theory of error and least square adjustment: Application in coordinate transformation

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Abstract

After detection and elimination of blunders, determination and correction of systematic errors the remaining random errors are adjusted by the method of least square. It is very useful technique in survey adjustments; as an example of its application, adjustment of conformal transformation of coordinate has been presented in this paper.

1. Basic concept of Errors

Geomatics engineers are usually faced with the problem of estimating some unknown quantities (parameters). This is done through collecting several measurements of some kind known as observations- and then adopting some appropriate mathematical model relating both observations and unknowns [Naser EL-Sheimy]

Observations generally require some form of instrumentation that is operated by some observer under certain environmental condition. Therefore, in every observation these three things- instrument, observer and environment have an influence on the accuracy of the measured quantity. Due to this influence, all observations contain error - the difference between an observation of a quantity and it true value.

The true value of a quantity is never known and hence the true error too. However, both quantities can be estimated.

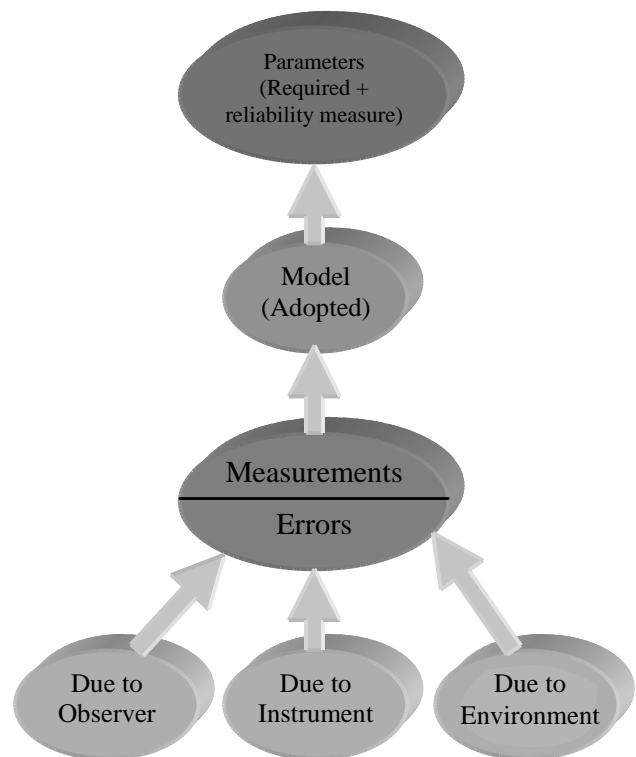$$\mathsf{e} = l - t$$

e : *true error*

l : *observed value*

t : *true value*

$$\mathsf{n} = l - \hat{x}$$

n : *estimated error*

l : *observed value*

$\hat{x}$: *estimated value*

Parameters (Required + reliability measure)

Model (Adopted)

Measurements

Errors

Due to Observer

Due to Instrument

Due to Environment

1.1 There are three sources of error

Personal: due to limitations in human perception power, carelessness, fatigue etc.

Instrumental: due to imperfect construction, improper and incomplete adjustment, precision limits (least count) etc.

Natural: due to changing environmental conditions, structure of the earth, gravity and gravitational forces, earth's rotation etc

	Gross errors, mistakes, or blunders:	**Systematic errors**	**Random errors**
Characteristics	These are the largest of the errors likely to arise/ abnormal observations	These can be constant or variable through out the observation and are generally attributable to a known circumstance. These can be expressed by some functional relationships.	These are those variants, which remain after all other errors have been removed. they have no functional relationship based up on a deterministic system usually modeled by probability theory
Source	Personal sources carelessness of the observer	All instrumental personal and natural sources can cause systematic error	All instrumental personal and natural sources can cause random error
Effect/ nature	Inhomogeneous observables	It is cumulative in nature. It shifts all the observations in a certain direction.	It is compensating in nature. Cannot generally be eliminated.
Treatment	Must be detected by careful checking and independent check measurements and eliminated by filtering out	Must be detected and corrected by proper calibration of the instrument before use. Use appropriate procedure during measurement, like reciprocal observations two face reading. Apply corrections.	Can be minimized by taking redundant observations and adjusting by method of least square. the process is referred to as the "Adjustment of observation" or "Adjustment computation"

Three types of errors

2. Principle of Least Square

As the true value and true error by the measurements of a quantity is never known, what do we do is we estimate the most probable value (MPV) which is supposed to be the nearest to the true value among or within the range of the large no of observations. The basic theorem in this subject is referred to as Gauss-Markov theorem, which states, "in the case of independent observations of equal weight the least square estimates are the linear unbiased estimates with minimum variance." In surveying literature the principle of least square is shown derived for the observations of normal distribution and is usually expressed as- "the most probable value or the best linear unbiased estimate of a set of observations is one for which the sum of weighted squared residuals is a minimum." [w. Schofield]

The expression for the normal distribution curve,

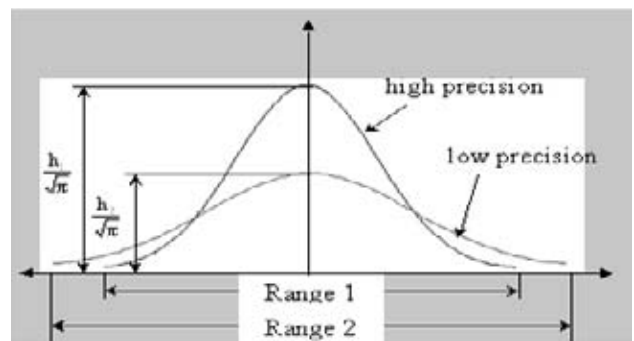the probability curve is given by $y = \dfrac{h}{\sqrt{\pi}} e^{-h^2 e^2}$

Figure 2: Probability Curve

Where

y -is the probability of occurrence of the error ε

h -is an index of precision

e -is the exponential function 2.71828

Differentiating with respect to *h* we will get

$$\frac{dy}{dh} = \frac{e^{-h^2\varepsilon^2}}{\sqrt{\pi}}\left(1 - 2h^2\varepsilon^2\right)$$

For a maximum *y*, $\frac{dy}{dh} = 0$

i.e. $1 - 2h^2\varepsilon^2 = 0$

or $\varepsilon^2 = \frac{1}{2h^2}$

Thus, the error will be decreased and the accuracy will be increased as the precision *h* increases the maximum accuracy will be achieved when

$$\sum_{i=1}^{n}\text{e}_i^{\ 2} = \sum_{i=1}^{n}\frac{1}{2h_i^{\ 2}} = \text{a minimum}$$

For weighted observations

$$\sum_{i=1}^{n}w_i\text{e}_i^{\ 2} = \text{a minimum}$$

ε_i = MPV (M) - ith observed value (M$_i$)
w_i = weight of ith observed value
Then,

$$\sum_{i=1}^{n}w_i\text{e}_i^{\ 2} = \sum_{i=1}^{n}w_i(M - M_i)^2 = \text{a minimum}$$

or $\quad \frac{d}{dM}\left(\sum_{i=1}^{n}w_i(M - M_i)^2\right) = 0$

or $\quad \sum_{i=1}^{n}w_i(M - M_i) = 0$

or $\quad \sum_{i=1}^{n}w_i(M - M_i) = 0$

or $\quad M\sum_{i=1}^{n}w_i - \sum_{i=1}^{n}w_iM_i = 0$

or $\quad M = \dfrac{\displaystyle\sum_{i=1}^{n}w_iM_i}{\displaystyle\sum_{i=1}^{n}w_i}$

thus the most probable value is the weighted arithmetic mean of the observed values. if the observed values are of equal weight i.e. $w_1=w_2=w_3 = ... = w_n = w$,

$$\sum_{i=1}^{n}w_i = w$$

and

$$\sum_{i=1}^{n}w_iM_i = w\sum_{i=1}^{n}M_i,$$

Then $M = \dfrac{\displaystyle\sum_{i=1}^{n}M_i}{n}$,

Thus for the observations with equal weight the most probable value is the simple arithmetic mean. There are two basic methods for adjustment of the observations by this technique, namely The Indirect Method or Method of Observation Equations, and Direct Method or The Method of Condition Equations.

In the Indirect Method, we will have following four steps:

Step 1: Assume the values for required quantities (the assumed value may be the observed quantity or any other arbitrarily assumed quantity)

Step 2: Formulate the observation equations by applying corrections (*v*s) to the assumed values to obtain MPV.

Step 3: Determine the residuals by subtracting observed values from MPV

Step 4: formulate the normal equations by applying least square conditions.

3. Adjustment of Conformal Transformation of Coordinates by Least Square Method

3.1. Conformal Transformation of Coordinates

Let us first derive the equations for the conformal transformation. A two Dimensional Conformal Transformation consists of three basic steps:

- Scale Change
- Rotation
- Translation

3.1.1 Scale Change

Let us say we have coordinate systems XY and EN as shown in figure locating the points a (X_a , Y_a) and b (X_b , Y_b) in XY system and their corresponding positions a (E_a , N_a) and b (E_b , N_b) in EN system, then the scale factor is given as
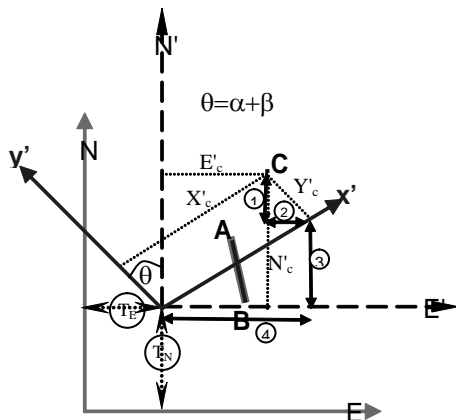
$$S = AB/ab \ = \sqrt{\frac{(E_b - E_a)^2 + (N_b - N_a)^2}{(X_b - X_a)^2 + (Y_b - Y_a)^2}}$$
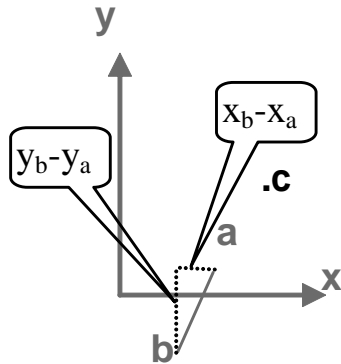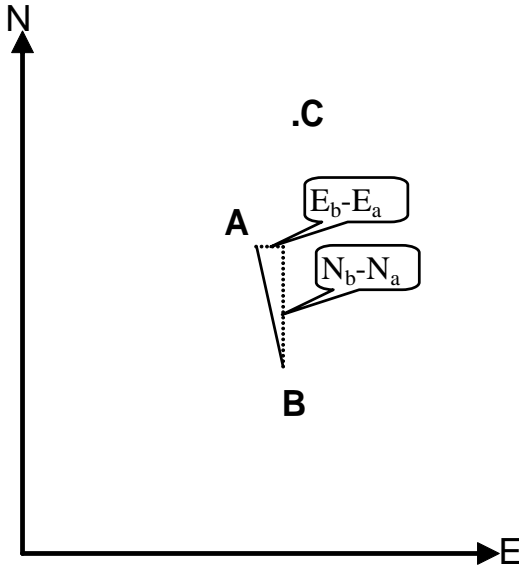
First, we will change the scale of XY system making equal to that of EN system by multiplying each x,y by the scale factor S . The scaled coordinates are x',y'.

3.1.2 Rotation

Here we super impose the scaled x' y' coordinate system over EN system, so that AB and ab of both the system coincide as shown in the figure below and construct an auxiliary axis system E'N' coinciding the origin with the origin (shifted during rotation) of x'y' system.

① $y'_c Cos\theta$ ③ $x'_c Sin\theta$
② $y'_c Sin\theta$ ④ $x'_c Cos\theta$

$$\alpha = Tan^{-1}\left(\frac{x_b - x_a}{y_b - y_a}\right)$$

$$\beta = Tan^{-1}\left(\frac{E_B - E_A}{N_B - N_A}\right)$$

$$E' = x'Cos\theta - y'Sin\theta$$
$$N' = x'Sin\theta + y'Cos\theta$$

3.1.3 Translation

It is the Translation from E'N' coordinate system to EN system

$$T_E = E_A - E'_A = E_B - E'_B$$
$$T_N = N_A - N'_A = N_B - N'_B$$

Finally we determine the coordinates in EN system as

$$E_c = T_E + E'_c = T_E + x'Cos\theta - y'Sin\theta$$
$$= T_E + S \times xCos\theta - S \times ySin\theta$$

$$N_c = T_N + N'_c = T_N + x'Sin\theta + y'Cos\theta$$
$$= T_N + S \times xSin\theta + S \times yCos\theta$$

if we suppose S$Sin\theta = b$ and S$Cos\theta = a$ we will have the general transformation equations as

$$E = T_E + ax - by$$
$$N = T_N + bx + ay$$

4. Transformation with Least Square Adjustment

If more than two control points are available, redundancy exists and the transformation can be computed using least square solution.

Let A, B and C are three control points known in both the XY and EN systems of coordinate.

As derived above, there will be 6 observation equations, two for each A, B and C, as follows: (which are with added residuals to make the redundant equations consistent.)

$$aX_A - bY_A + T_E = E_A + v_{E_A}$$
$$aY_A + bX_A + T_N = N_A + v_{N_A}$$
$$aX_B - bY_B + T_E = E_B + v_{E_B}$$
$$aY_B + bX_B + T_N = N_B + v_{N_B}$$
$$aX_C - bY_C + T_E = E_C + v_{E_C}$$
$$aY_C + bX_C + T_N = N_C + v_{N_C}$$

For least square adjustment, let us square and add the residuals

$$\sum v^2 = (aX_A - bY_A + T_E - E_A)^2$$
$$+ (aY_A + bX_A + T_N - N_A)^2$$
$$+ (aX_B - bY_B + T_E - E_B)^2$$
$$+ (aY_B + bX_B + T_N - N_B)^2$$
$$+ (aX_C - bY_C + T_E - E_C)^2$$
$$+ (aY_C + bX_C + T_N - N_C)^2$$

If we take the partial derivative of the $\sum v^2$ with respect to a, b, T_E and T_N and compare each with zero to fulfill the condition of minima we will get four equations that can be represented by the following matrix equation.

$$\frac{\partial}{\partial a}\sum v^2 = 0, \qquad \frac{\partial}{\partial b}\sum v^2 = 0,$$
$$\frac{\partial}{\partial T_E}\sum v^2 = 0, \quad \text{and} \quad \frac{\partial}{\partial T_N}\sum v^2 = 0$$

$$A^T A X = A^T L$$

Where,

$$A = \begin{pmatrix} X_A & -Y_A & 1 & 0 \\ Y_A & X_A & 0 & 1 \\ X_B & -Y_B & 1 & 0 \\ Y_B & X_B & 0 & 1 \\ X_C & -Y_C & 1 & 0 \\ Y_C & X_C & 0 & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} a \\ b \\ T_E \\ T_N \end{pmatrix}, L = \begin{pmatrix} E_A \\ N_A \\ E_B \\ N_B \\ E_C \\ N_C \end{pmatrix}$$

The solution of the equation
$A^T A X = A^T L$ is
$$X = (A^T A)^{-1} A^T L$$
We can get $A^T A$ from A as

$$(A^T A) = \begin{pmatrix} \sum(X_i^2 + Y_i^2) & 0 & \sum X_i & \sum Y_i \\ 0 & \sum(X_i^2 + Y_i^2) & -\sum Y_i & \sum X_i \\ \sum X_i & -\sum Y_i & 3 & 0 \\ \sum Y_i & \sum X_i & 0 & 3 \end{pmatrix}$$

$$i = A, B, C$$

If we assume $\alpha = \sum(X_i^2 + Y_i^2)$,
$\beta = \sum X_i$ and $\gamma = \sum Y_i$

the matrix will be formed as

$$(A^T A) = \begin{pmatrix} \alpha & 0 & \beta & \gamma \\ 0 & \alpha & -\gamma & \beta \\ \beta & -\gamma & 3 & 0 \\ \gamma & \beta & 0 & 3 \end{pmatrix}$$

Calculating determinant of the matrix we will get
$$Det(A^T A) = \psi^2,$$
$$where, \psi = (3\alpha - \beta^2 - \gamma^2)$$

Again determining the inverse matrix we will get

$$(A^T A)^{-1} = \frac{1}{\psi^2}\begin{pmatrix} 3\psi & 0 & -\beta\psi & -\gamma\psi \\ 0 & 3\psi & \gamma\psi & -\beta\psi \\ -\beta\psi & \gamma\psi & \alpha\psi & 0 \\ -\gamma\psi & -\beta\psi & 0 & \alpha\psi \end{pmatrix}$$

If we assume
$3/\psi = p, -\beta/\psi = q, -\gamma/\psi = r$
and $\alpha/\psi = s$ we will have

$$(A^T A)^{-1} = \begin{pmatrix} p & 0 & q & r \\ 0 & p & -r & q \\ q & -r & s & 0 \\ r & q & 0 & s \end{pmatrix}$$

Again, we can get $A^T L$ from A and L as follows

$$A^T L = \begin{pmatrix} \sum(X_i E_i + Y_i N_i) \\ \sum(X_i N_i - Y_i E_i) \\ \sum E_i \\ \sum N_i \end{pmatrix} = \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}(say)$$

$$i = A, B, C$$

If we operate $A^T L$ with $(A^T A)^{-1}$ we will get X

$$X = \begin{pmatrix} a \\ b \\ T_E \\ T_N \end{pmatrix} = (A^T A)^{-1} A^T L = \begin{pmatrix} p & 0 & q & r \\ 0 & p & -r & q \\ q & -r & s & 0 \\ r & q & 0 & s \end{pmatrix} \times \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

Finally, we will get the adjusted transformation parameters as

$$\begin{pmatrix} a \\ b \\ T_E \\ T_N \end{pmatrix} = \begin{pmatrix} pe + qg + rh \\ pf - rg + qh \\ qe - rf + sg \\ re + qf + sh \end{pmatrix}$$

Getting these parameters, we can transform any point D in XY coordinate system to EN coordinate system as

$$E_D = aX_D - bY_D + T_E$$
$$N_D = aY_D + bX_D + T_N$$

5. Conclusion and Recommendations

The theory of error and the method of Least Square adjustment has been presented in brief in this paper, the mathematical computation for the Least Square adjustment seems to be rather cumbersome if we proceed manually, it is recommended to use computer program to make its execution easier and faster. During the adjustment of survey observations or computations, one should first detect the blunders. Those observations which are larger than 3σ i.e. E99.9 are assumed to be blunders, first they should be detected filtered out. After that the corrections to the systematic errors and biases should be applied. And finally, only the remaining random errors should be adjusted by applying the method of Least Square Adjustment.

References

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