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Analysis of two hybrid schemes to solve the Benjamin-Bona-Mahony (BBM) equation

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Abstract

In this research, we compare and contrast two numerical methods, the Laplace Adomian Decomposition Method (LADM) and the Elzaki Projected Differential Transform Method (EPDTM), for solving the Benjamin-Bona-Mahony (BBM) equation, and this helps in evaluating their performance, assessing robustness, gaining insights into solution behavior, validating results, generalizing their applicability, and advancing the field. This rigorous comparisons validates the efficacy of the methods as we compare the exact results and the results of LADM and EPDTM, plotting the convergent of the two methods to identify the method that is highly convergent and accurate. Results identify the Elzaki Projected Differential Transform Method (EPDTM) as having a simple and accurate numerical method for solving the BBM equation. EPDTM enhances accessibility, computational efficiency, accuracy, robustness, and validation of the solutions, supporting the understanding and analysis of wave phenomena described by the BBM equation. This study contributes to the advancement of the field of numerical analysis and computational science as practical applications of BBM equation in various fields, including coastal engineering, wave modeling, and wave stability analysis cannot be over emphasize.

Keywords: Benjamin-Bona-Mahony equation; Laplace Adomian Decomposition Method; Elzaki Projected Differential Transform Method.

1. Introduction

Differential equations serve as a crucial mathematical tool for modeling, analyzing, and predicting the behavior of dynamic systems in numerous scientific, engineering, and mathematical applications [1-4]. They bridge theoretical understanding with practical applications, enabling us to solve complex problems, make predictions, and advance our understanding of the natural and engineered world [5-8].

Nonlinear PDEs are crucial for modeling and understanding complex phenomena, capturing rich dynamics and behaviors, stability analysis, simulating real-world systems, and advancing mathematical theory [1, 9-11]. Researchers have make use of this to a more accurate representation of many physical and mathematical systems such as the Schrodinger equation governing wave-duality in quantum physics and optics, Allen-Cahn equation governing oil pollution dynamics in oceanography [12], fisher's reaction-diffusion Eq. [13-16] and they have been offering a deeper insight into their behavior compared to linear PDEs.

Complex physical, biological, and engineering events are frequently modeled with nonlinear PDEs. The behavior of many real-world situations is nonlinear, while linear PDEs offer streamlined approximations [11, 17, 18]. For complex systems to be effectively modeled, nonlinear PDEs must be able to capture the nonlinear linkages, interactions, and feedback processes. Examples include nonlinear reaction-diffusion processes, turbulent fluid flow, pattern generation, and nonlinear wave propagation [19-22].

Nonlinear PDEs are frequently employed to investigate emergent behavior in systems with nonlinear interactions. Nonlinearities can lead to the emergence of new phenomena, such as solitons, shocks, instabilities, and pattern formation [7, 21-25]. By studying the solutions of nonlinear PDEs, researchers gain insights into the underlying mechanisms responsible for the emergence of these complex and fascinating behaviors [14, 17, 26, 27].

Researchers have used numerical techniques to solve nonlinear differential equations in a practical and efficient manner over the years. They offer a versatile, effective, and dependable method for approximating solutions, comprehending complex dynamics, dealing with complex geometries, and investigating the behavior of nonlinear systems. Although approximate, numerical method solutions can offer useful information and aid in decision-making in a variety of scientific, engineering, and mathematical applications [9, 28, 29]. Researchers continue to implement new numerical methods for solving nonlinear PDEs to improve accuracy, efficiency, robustness, adaptability, and applicability to complex problems. These advancements in numerical methods are driven by the need to tackle challenging nonlinear systems, exploit specific mathematical structures, optimize criteria, handle multiscale or multiphysics problems, leverage software and implementation advancements, and validate against existing methods [3, 19, 30, 31]. Through these efforts, researchers aim to advance state-of-the-art numerical techniques and provide more reliable and accurate solutions for a wide range of nonlinear PDE problems [28, 32, 33].

The Laplace-Adomian Decomposition Method is one of the numerical method that offers an alternative approach for solving nonlinear PDEs by combining the Laplace transform and the Adomian decomposition method. It provides an approximate solution that can be refined by including more terms in the series [16, 21, 24, 34, 35]. However, it is important to note that the LADM may have limitations for certain types of nonlinearities and complex PDEs, and its convergence and accuracy should be carefully assessed for each problem and that is focus of this research.

It is important to note that the Elzaki Transform Method is one

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of many available techniques for solving PDEs. Its effectiveness, accuracy, and convergence depend on the nature of the problem and the choice of decomposition functions. The Projected Differential Transform Method provides an alternative approach for approximating solutions to nonlinear PDEs by combining the DTM with a projection technique. It offers a systematic and iterative framework for obtaining approximate solutions, reducing the PDE to a set of algebraic equations. Researchers have combined Elzaki and Projected Differential Transform Method and also have combined Laplace Adomian Decomposition to solve a couple of nonlinear differential equations [4, 36-38].

Understanding the behavior of waves and their interactions is crucial for various applications in coastal and ocean engineering, marine science, and related fields. The BBM equation, with its capability to capture dispersion and nonlinear effects, provides insights into wave propagation, wave breaking, and wave-structure interactions. This knowledge aids in designing coastal structures, predicting wave behavior, and mitigating the impacts of waves on coastal communities and ecosystems. The Benjamin-Bona-Mahony (BBM) equation is a nonlinear partial differential equation that arises in the study of water waves and fluid dynamics. It is a modified version of the classic Korteweg-de Vries (KdV) equation, incorporating additional terms that account for dispersion effects and higher-order nonlinearities [15, 39-53].

Although many methods have been used to find the analytical solution of the BBM equation but the uniqueness of this research is that we compare and contrast two numerical methods, the Laplace Adomian Decomposition Method (LADM) and the Elzaki Projected Differential Transform Method (EPDTM), for solving the Benjamin-Bona-Mahony (BBM) equation, and this helps in evaluating their performance, assessing robustness, gaining insights into solution behavior, validating results, generalizing their applicability, and advancing the field. It aids researchers in selecting appropriate methods, improving existing techniques, and expanding the knowledge base in numerical analysis and PDE solving.

2. Materials and method

2.1. General solution to Benjamin-Bona-Mahony equation using Laplace Adomian Decomposition

The Benjamin-Bona-Mahony (BBM) equation is a nonlinear partial differential equation that describes the propagation of long water waves. The general form of the BBM equation is:

$$u_t(x,t) + u_x(x,t) + u(x,t)u_x(x,t) - u_{xxt}(x,t) = 0 \tag{1}$$

with initial condition given by

$$u(x,0) = g(x) \tag{2}$$

We re-write Eq. (1) as

$$u_t(x,t) = u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t) \tag{3}$$

Taking the Laplace transform of both sides of the given BBM Eq. (1), we have

$L\left[u_t\left(x,t\right)\right] = L\left[u\left(x,t\right)u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]$, and this gives

$sU(x,s) - u(x,0) = L\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]$

Simplifying further we have

$$sU(x,s) = u(x,0) + L\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \tag{4}$$

Substituting the initial condition(2) in above Eq. (4) we have

$sU(x,s) = g(x) + L\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]$

By simplification, we have

$$u(x,s) = \frac{g(x)}{s} + \frac{L}{s}\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \tag{5}$$

Taking the inverse Laplace transform of above Eq. (5), we have

$$L^{-1}\left[u\left(x,s\right)\right] = L^{-1}\left[\frac{g(x)}{s}\right] + L^{-1}\left[\frac{L}{s}\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right]$$

Then

$$u(x,t) = g(x) + L^{-1}\left[\frac{L}{s}\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right] \tag{6}$$

We take the unknown function and the infinite series solution as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Also, the nonlinear term in Eq. (6) can be decomposed as

$$u(x,t)u_x(x,t) = \sum_{n=0}^{\infty} A_n(x,t)$$

And

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{\infty} \lambda^i u_{kx}\right)\right]_{\lambda=0}$$

are Adomian polynomials.

Finding first few of the Adomian polynomials we have

$A_0 = u_0(x,t)u_{0x}(x,t)$, $A_1 = u_0(x,t)u_{1x}(x,t) + u_1(x,t)u_{0x}(x,t)$,

And so on. This implies that

$$\sum_{n=0}^{\infty} u_n(x,t) = g(x) + L^{-1}\left[\frac{L}{s}\left[\sum_{n=0}^{\infty} u_{nxxt}(x,t) - \sum_{n=0}^{\infty} u_{nx}(x,t) - \sum_{n=0}^{\infty} A_n\right]\right]$$

The recursive relation is as follow

$$u_0(x,t) = g(x) \tag{7}$$

$, u_1(x,t) = L^{-1}\left[\frac{L}{s}\left[u_{0xxt}(x,t) - u_{0x}(x,t) - A_0(x,t)\right]\right],$

$u_2(x,t) = L^{-1}\left[\frac{L}{s}\left[u_{1xxt}(x,t) - u_{1x}(x,t) - A_1(x,t)\right]\right],$

....

....

....

$u_{n+1}(x,t) = L^{-1}\left[\frac{L}{s}\left[u_{nxxt}(x,t) - u_{nx}(x,t) - A_n(x,t)\right]\right].$

Computing the first term of the solution, we have

$u_0(x,t) = g(x),$

$$\begin{aligned} u_1(x,t) &= L^{-1}\left[\frac{L}{s}\left[0-g^{'}(x)-g(x)g^{'}(x)\right]\right] \\ &= L^{-1}\left[\frac{L}{s}\left[-g^{'}(x)-g(x)g^{'}(x)\right]\right] \\ &= L^{-1}\left[\frac{L}{s}\left[-g^{'}(x)(1+g(x))\right]\right] \\ &= L^{-1}\left[\frac{1}{s}\left[-g^{'}(x)(1+g(x))\right]L(1)\right] \\ &= L^{-1}\left[\frac{1}{s}\left[-g^{'}(x)(1+g(x))\right]\frac{1}{s}\right] \\ &= \left[L^{-1}\left(\frac{1}{s^2}\right)\left[-g^{'}(x)(1+g(x))\right]\right] \\ &= -g^{'}(x)(1+g(x))\left[L^{-1}\left(\frac{1}{s^2}\right)\right] \end{aligned}$$

$\therefore u_1(x,t) = -g^{'}(x)(1+g(x))t.$

The other terms can be computed in the same similar way.

The final solution is given as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+... \tag{8}$$

2.2. General solution to Benjamin-Bona-Mahony equation using Elzaki Projected Differential Transform Method

From the given BBM Eq. (1) and initial condition (2)

$u_t(x,t) + u_x(x,t) + u(x,t)u_x(x,t) - u_{xxt}(x,t) = 0$

$u(x,0) = g(x)$

Taking the Elzaki Transform of the BBM equation, it gives

$$E\left[u_t\left(x,t\right)\right] = E\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \tag{9}$$

Now, applying the differentiation property of Elzaki transform to get

$$E\left[\frac{u(x,t)}{v} - vu(x,0)\right] = E\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]$$

By simplification we have

$$E\left[u\left(x,t\right)\right] = v^2u(x,0) + vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]$$

Put $u(x,0) = g(x)$, we have

$$E\left[u\left(x,t\right)\right] = v^2g(x) + vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]$$

Now, taking the inverse Elzaki transform we get:

$$E\left\{E\left[u\left(x,t\right)\right]\right\} = E^{-1}\left\{vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right\}$$

$$E\left\{E\left[u\left(x,t\right)\right]\right\} = E^{-1}\left\{v^2g(x)\right\} + E^{-1}\left\{vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right\}$$

Then,

$$u(x,t) = g(x) + E^{-1}\left\{vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right\} \tag{10}$$

$g(x)$ is the first of the series and also the prescribed initial condition, to get the series of Eq. (9) we used the general form and applied the PDTM as shown below.

$$\sum_{k=0}^{\infty} u(x,k+1) = E^{-1}\left\{vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right\} \tag{11}$$

By applying the PDTM, we have

Then,$u(x,k+1) = E^{-1}\left\{vE\left[A_k - B_k - C_k\right]\right\}$;

Note, $A_k, B_k, and\ C_k$ are the projected differential transform of $u_{xxt}(x,t), u_x(x,t)\ andu(x,t)u_x(x,t)$ respectively.

Taking $k=0$

$$u(x,1) = E^{-1}\left[vE\left[A_0 - B_0 - C_0\right]\right]$$

For $k=1$

$$u(x,2) = E^{-1}\left[vE\left[A_1 - B_1 - C_1\right]\right]$$

For $k=2$

$$u(x,3) = E^{-1}\left[vE\left[A_2 - B_2 - C_2\right]\right]$$

Similarly, we can get the other series terms for $k = n$, which gives us

$$u(x,n+1) = E^{-1}\left[vE\left[A_n - B_n - C_n\right]\right]$$

With this we have the general approximate solution for Elzaki Projected Differential Transform method to be

$$u(x,t) = \sum_{k=0}^{\infty} u(x,k)$$

$$u(x,t) = u(x,1) + u(x,2) + u(x,3) + u(x,4) + ... \tag{12}$$

3. Study 1: Solution of BBM equation(1) with initial conditions given by $u(x,0) = x$

3.1. Using Laplace Adomian Decomposition

$$u_t(x,t) + u_x(x,t) + u(x,t)u_x(x,t) - u_{xxt}(x,t) = 0 \tag{13}$$

With $u(x,0) = x$

Simplifying Eq. (12) and taking the Laplace transform of the both sides, we have

$$sU(x,s) = u(x,0) + L\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \tag{14}$$

Substituting the initial condition $u(x,0) = x$ in above Eq. (13) we have

$$U(x,s) = \frac{x}{s} + \frac{L}{s}\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \tag{15}$$

Taking the inverse Laplace transform of above Eq. (14), we have the recursive relation

$$u(x,t) = x + L^{-1}\left[\frac{L}{s}\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right] \tag{16}$$

Here we have our $u(x,0) = x$
Take the unknown function and the infinite series solution has

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{17}$$

Also, the nonlinear term $u(x,t)u_x(x,t)$ in Eq. (15) can be decomposed as

$$u(x,t)u_x(x,t) = \sum_{n=0}^{\infty} A_n(x,t) \tag{18}$$

And

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_{kx}\right)\right]_{\lambda=0}$$

are Adomian polynomials.
And so on. Then, using the recursive relation described above (15) to compute few terms of the solution, we have

$$u_{n+1}(x,t) = L^{-1}\left[\frac{L}{s}\left[\sum_{n=0}^{\infty} u_{nxxt}(x,t) - \sum_{n=0}^{\infty} u_{nx}(x,t) - \sum_{n=0}^{\infty} A_n\right]\right]$$

For $n = 0$, we have

$$\begin{aligned} A_0 &= \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[\left(\sum_{i=0}^{0}\lambda^i u_i\right)\left(\sum_{i=0}^{0}\lambda^i u_{kx}\right)\right]_{\lambda=0} \\ &= u_0(x,t)u_{0x}(x,t) = x.1 = x \end{aligned} \tag{19}$$

Then the solution for $u_1(x,t)$ is given as

$$u_1(x,t) = L^{-1}\left[\frac{L}{s}\left[u_{0xxt}(x,t) - u_{0x}(x,t) - A_0(x,t)\right]\right]$$

$$\begin{aligned} u_1(x,t) &= L^{-1}\left[\frac{L}{s}\left[[x]_{xxt} - [x]_x - A_0(x,t)\right]\right] \\ &= L^{-1}\left[\frac{L}{s}\left[0 - 1 - x\right]\right] \end{aligned}$$

$$L^{-1}\left[(-1-x)\frac{1}{s}L[1]\right] = (-1-x)L^{-1}\left[\frac{1}{s}\right]$$

$$u_1(x,t) = (-1-x)t$$

For $n = 1$, we have

$$\begin{aligned} A_1 &= \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[\left(\sum_{i=0}^{0}\lambda^i u_i\right)\left(\sum_{i=0}^{0}\lambda^i u_{kx}\right)\right]_{\lambda=0} \\ &= u_0(x,t)u_{1x}(x,t) + u_1(x,t)u_{0x}(x,t) \\ &= (-2x-1)t \end{aligned} \tag{20}$$

Then the solution for $u_2(x,t)$ is given as

$$u_2(x,t) = L^{-1}\left[\frac{L}{s}\left[u_{1xxt}(x,t) - u_{1x}(x,t) - A_0(x,t)\right]\right]$$

$$u_2(x,t) = L^{-1}\left[\frac{L}{s}\left[[(-1-x)]_{xxt} - [(-1-x)]_x - (-2x-1)t\right]\right]$$

$$\begin{aligned} &= L^{-1}\left[\frac{L}{s}\left[0 + t + (2x+1)t\right]\right] \\ &= L^{-1}\left[(2+2x)\frac{1}{s}L[t]\right] \\ &= (2+2x)L^{-1}\left[\frac{1}{s^2}\right] \\ u_2(x,t) &= (2+2x)\frac{t^2}{2} \\ u_2(x,t) &= (1+x)t^2 \end{aligned} \tag{21}$$

For $n = 2$, we have

$$A_2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_{kx}\right)\right]_{\lambda=0} \tag{22}$$

$$\frac{1}{2}\left(u_0(x,t)u_{2x}(x,t) + u_1(x,t)u_{1x}(x,t) + u_2(x,t)u_{0x}(x,t)\right)$$

$$= \frac{1}{2}\left\{(2+3x)t^2\right\}$$

$$A_2 = (2+3x)\frac{t^2}{2}$$

Then the solution for $u_3(x,t)$ is given as

$$u_3(x,t) = L^{-1}\left[\frac{L}{s}\left[u_{2xxt}(x,t) - u_{2x}(x,t) - A_2(x,t)\right]\right]$$

$$= L^{-1}\left[\frac{L}{s}\left[[(1+x)t^2]_{xxt} + [(1+x)t^2]_x - (2+3x)\frac{t^2}{2}\right]\right]$$

$$= L^{-1}\left[\frac{L}{s}\left[\left(-\left(2+\frac{3}{2}x\right)\right)t^2\right]\right]$$

$$u_3(x,t) = L^{-1}\left[\left[-\left(2+\frac{3}{2}x\right)\frac{1}{s}L(t^2)\right]\right] = L^{-1}\left[-\left(2+\frac{3}{2}x\right)\frac{1}{s}.\frac{2}{s^3}\right] = \left[-\left(2+\frac{3}{2}x\right)L^{-1}\left(\frac{2}{s^4}\right)\right]$$

$$\begin{aligned} u_3(x,t) &= -\left(2+\frac{3}{2}x\right)\frac{2t^3}{3!} \\ &= -\left(2+\frac{3}{2}x\right)\frac{t^3}{3} \end{aligned} \tag{23}$$

Finally, the approximate solution is given as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots$$

$$u(x,t) = x - (1+x)t + (1+x)t^2 - \left(2+\frac{3}{2}x\right)t^3 + \dots \tag{24}$$

3.2. Using Elzaki Projected Differential Transform Method

$$u_t(x,t) + u_x(x,t) + u(x,t)u_x(x,t) - u(x,t)u_{xxt}(x,t) = 0 \tag{25}$$

With $u(x,0) = x$
Taking the Elzaki Transform of Eq. (25), it gives

$$E\left[u_t(x,t)\right] = E\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \tag{26}$$

Now, applying the differentiation property of Elzaki transform to get

$$E\left[u(x,t)\right] = v^2 u(x,0) + vE \left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \quad (27)$$

Put $u(x,0) = x$, we have

$$E\left[u(x,t)\right] = x + vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \quad (28)$$

Now, taking the inverse Elzaki transform we get:

$$E^{-1}\left\{E\left[u(x,t)\right]\right\} = E^{-1}\left\{v^2 x\right\} + E^{-1} \left\{vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right\}$$

Then,

$$u(x,t) = x + E^{-1}\left\{vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right\} \quad (29)$$

And the recursive relation for the Eq. (29)

$$\sum_{k=0}^{\infty} u(x,k+1) = E^{-1} \left\{vE\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right\} \quad (30)$$

By applying the PDTM, we have

$$u(x,k+1) = E^{-1}\left\{vE[A_k - B_k - C_k]\right\} \quad (31)$$

Note, $A_k = [u(x,k)]_{xxt}$, $B_k = [u(x,k)]_x$, and $C_k = \sum_{n=0}^{k} u(x,n)(u(x,k-n))_x$ are the projected differential transform of $u_{xxt}(x,t)$, $u_x(x,t)$, and $u(x,t)u_x(x,t)$ respectively.

Taking $k = 0$

$$u(x,1) = E^{-1}\left[vE[A_0 - B_0 - C_0]\right] \quad (32)$$

$A_0 = [u(x,0)]_{xxt} = (x)_{xxt} = 0;$
$B_0 = [u(x,0)]_x = (x)_x = 1;$
$C_0 = \sum_{n=0}^{k=0} u(x,n)(u(x,k-n))_x = u(x,0)(u(x,0))_x = x.1 = x$
$u(x,1) = E^{-1}\left\{vE[0-1-x]\right\} = E^{-1}\left\{v[-1-x]E(1)\right\} = E^{-1}\left\{[-1-x]vv^2\right\}$
$= \left\{[-1-x]\, E^{-1}\left(v^3\right)\right\} = [-1-x]\, t$

Then,

$$u(x,1) = -(1+x)t \quad (33)$$

For $k = 1$

$$u(x,2) = E^{-1}\left[vE[A_1 - B_1 - C_1]\right] \quad (34)$$

$A_1 = [u(x,1)]_{xxt} = (-1(1+x)t)_{xxt} = 0;$
$B_1 = [u(x,1)]_x = (-(1+x)t)_x = t;$

$$\begin{aligned} C_1 &= \sum_{n=0}^{k=1} u(x,n)(u(x,k-n))_x \\ &= u(x,0)(u(x,1))_x + u(x,1)(u(x,0))_x \\ &= x.\left(-(1+x)t\right)_x - (1+x)t.(x)_x \\ &= -(1+2x)t \end{aligned}$$

So we have

$$u(x,2) = E^{-1}\left[vE[0 + (1+x)t + (1+2x)t]\right] \quad (35)$$

Then,

$$u(x,2) = E^{-1}\left[vE[(2+2x)t]\right]$$

$$= E^{-1}\left\{[v(2+2x)E(t)]\right\} = E^{-1}\left\{\left[(2+2x)v.v^3\right]\right\}$$

$$= (2+2x)E^{-1}(v^4) = (2+2x)\frac{t^2}{2!} = (1+x)t^2$$

$$\therefore u(x,2) = (1+x)t^2 \quad (36)$$

For $k = 2$

$$u(x,3) = E^{-1}\left[vE[A_2 - B_2 - C_2]\right] \quad (37)$$

$A_2 = [u(x,2)]_{xxt} = \left((1+x)t^2\right)_{xxt} = 0;$
$B_2 = [u(x,2)]_x = \left((1+x)t^2\right)_x = t^2;$

$$\begin{aligned} C_2 &= \sum_{n=0}^{k=2} u(x,n)(u(x,k-n))_x))_x \\ &= u(x,0)(u(x,2))_x + u(x,1)(u(x,1))_x + u(x,2)(u(x,0 \\ &= x.\left((1+x)t^2\right)_x + (-1-x)t.((-1-x)t)_x \\ &\quad + \left((1+x)t^2\right).(x)_x \\ &= (2+3x)t^2 \end{aligned}$$

So we have,

$$u(x,3) = E^{-1}\left[vE[0 - t^2 - (2+3x)t^2]\right] \quad (38)$$

$$\begin{aligned} u(x,3) &= E^{-1}\left[vE\left[-(3+3x)t^2\right]\right] \\ &= E^{-1}\left[-3(3+3x)vE(t^2)\right] = E^{-1}\left[-3(3+3x)v.2!.v^4\right] \\ &= E^{-1}\left[-3(3+3x)2!.v^5\right] = -(3+3x).2!E^{-1}(v^5) \end{aligned}$$

$$u(x,3) = -(1+x)3.\frac{2!}{3!}.t^3$$

$$u(x,3) = -(1+x).t^3 \quad (39)$$

And so on, then the solution of the BBM equation using EPDTM with initial condition $u(x,0) = x$ is given as,

$$u(x,t) = x - (1+x)t + (1+x)t^2 - (1+x)t^3 + ... \quad (40)$$

The above series solution converges to a replica of the exact solution using a Mathematical computation package

$$u(x,t) = \frac{x-t}{1+t}$$

4. Study 2: Solution of BBM equation(1) with initial conditions given by $u(x,0) = sech^2\left(\frac{x}{4}\right)$

4.1. Using Laplace Adomian Decomposition

$$u_t(x,t) + u_x(x,t) + u(x,t)u_x(x,t) - u_{xxt}(x,t) = 0 \quad (41)$$

With $u(x,0) = sech^2\left(\frac{x}{4}\right)$

Simplify Eq. (41) and taking the Laplace transform of the both sides, we have

$$sU(x,s) = u(x,0) + L\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \quad (42)$$

Substituting the initial condition $u(x,0) = sech^2\left(\frac{x}{4}\right)$ in above Eq. (42) and simplify, we have

$$U(x,s) = \frac{sech^2\left(\frac{x}{4}\right)}{s} + \frac{L}{s}\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right] \quad (43)$$

Taking the inverse Laplace transform of above Eq. (43), and with simplification, we have

$$u(x,t) = sech^2\left(\frac{x}{4}\right) + L^{-1}\left[\frac{L}{s}\left[u_{xxt}(x,t) - u_x(x,t) - u(x,t)u_x(x,t)\right]\right] \quad (44)$$

Here we have our $u(x,0) = sech^2\left(\frac{x}{4}\right)$

And

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_{kx}\right)\right]_{\lambda=0}$$

are Adomian polynomials.

Also our recurrence relation is given as

$$u_{n+1}(x,t) = L^{-1}\left[\frac{L}{s}\left[u_{nxxt}(x,t) - u_{nx}(x,t) - A_n(x,t)\right]\right] \quad (45)$$

For $n = 0$, and following the procedure of study 1, we have

$$A_0 = \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[\left(\sum_{i=0}^{0}\lambda^i u_i\right)\left(\sum_{i=0}^{0}\lambda^i u_{kx}\right)\right]_{\lambda=0} = u_0(x,t)u_{0x}(x,t)$$

$$A_0 = -\frac{1}{2}sech^2\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right) \quad (46)$$

Substituting the initial condition $u(x,0) = sech^2\left(\frac{x}{4}\right)$ and A_0 in our recurrence relation (45), we have

$$u_1(x,t) = L^{-1}\left[\frac{L}{s}\left[u_{0xxt}(x,t) - u_{0x}(x,t) - A_0(x,t)\right]\right]$$

$$= \left(\frac{1}{2}sech^2\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right) + \frac{1}{2}sech^4\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right)\right).t \quad (47)$$

For $n = 1$, we have

$$\begin{aligned} A_1 &= \frac{1}{1!}\frac{d}{d\lambda}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_{kx}\right)\right]_{\lambda=0} \\ &= u_0(x,t)u_{1x}(x,t) + u_1(x,t)u_{0x}(x,t), \end{aligned}$$

$$A_1 = -\frac{1}{8}\frac{\left(4cosh^4\left(\frac{x}{4}\right) + cosh^2\left(\frac{x}{4}\right) - 7\right)t}{cosh^8\left(\frac{x}{4}\right)} \quad (48)$$

Also, by substituting the $u_1(x,t)$ and A_1 in our recurrence relation (45), we have

$$u_2(x,t) = L^{-1}\left[\frac{L}{s}\left[u_{1xxt}(x,t) - u_{1x}(x,t) - A_1(x,t)\right]\right]$$

$$u_2(x,t) = \frac{1}{16}\frac{1}{cosh^8\left(\frac{x}{4}\right)}\begin{pmatrix} \left(2sinh\left(\frac{x}{4}\right)cosh^5\left(\frac{x}{4}\right) - 15sinh\left(\frac{x}{4}\right)cosh\left(\frac{x}{4}\right)\right)t \\ \left(-8cosh^2\left(\frac{x}{4}\right) + 4cosh^6\left(\frac{x}{4}\right) + 10cosh^4\left(\frac{x}{4}\right) - 14\right)\frac{t^2}{2} \end{pmatrix}$$

For $n = 2$, we have

$$\begin{aligned} A_2 &= \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\left(\sum_{i=0}^{\infty}\lambda^i u_{kx}\right)\right]_{\lambda=0} \\ &= \frac{1}{2}\left(u_0(x,t)u_{2x}(x,t) + u_1(x,t)u_{1x}(x,t) + u_2(x,t)u_{0x}(x,t)\right) \end{aligned}$$

$$= \frac{1}{2}\left\{-\frac{1}{64}\frac{1}{cosh^{11}\left(\frac{x}{4}\right)}\left(t\left(\begin{array}{c} -90sinh\left(\frac{x}{4}\right)t + 16tsinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right) + 42tsinh\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right) \\ -48tsinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right) + 8cosh^7\left(\frac{x}{4}\right) - 134cosh^3\left(\frac{x}{4}\right) \\ +2cosh^5\left(\frac{x}{4}\right) + 135cosh\left(\frac{x}{4}\right) \end{array}\right)\right)\right\}$$

$$A_2 = -\frac{1}{128}\frac{1}{cosh^{11}\left(\frac{x}{4}\right)}\left(\begin{array}{c} \left(\begin{array}{c} 90sinh\left(\frac{x}{4}\right) + 16sinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right) + 42sinh\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right) \\ -48sinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right) \end{array}\right)t^2 \\ \left(\begin{array}{c} 8cosh^7\left(\frac{x}{4}\right) - 134cosh^3\left(\frac{x}{4}\right) \\ +2cosh^5\left(\frac{x}{4}\right) + 135cosh\left(\frac{x}{4}\right) \end{array}\right)t \end{array}\right)$$

Also, by substituting the $u_2(x,t)$ and A_2 in our recurrence relation (45), we have

$$u_3(x,t)=\frac{1}{128}\frac{1}{cosh^{11}\left(\frac{x}{4}\right)}\left\{\begin{array}{c}\left(\begin{array}{c}4sinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right)+4sinh\left(\frac{x}{4}\right)cosh^8\left(\frac{x}{4}\right)-300sinh\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right)\\+420sinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right)\end{array}\right)t\\ \left(-422cosh\left(\frac{x}{4}\right)+16cosh^9\left(\frac{x}{4}\right)+80cosh^7\left(\frac{x}{4}\right)-204cosh^3\left(\frac{x}{4}\right)+639cosh\left(\frac{x}{4}\right)\right)\frac{t^2}{2}\\ \left(\begin{array}{c}8sinh\left(\frac{x}{4}\right)cosh^8\left(\frac{x}{4}\right)+56sinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right)-6sinh^6\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right)\\-160sinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right)-90sinh\left(\frac{x}{4}\right)\end{array}\right)\frac{2t^3}{6}\end{array}\right\}$$

Finally, the approximate solution is given as

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+... \tag{49}$$

Substituting the solution of $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t)$ and the initial condition in (49), we have the solution of BBM equation using LADM.

$$\begin{aligned}u(x,t) \;=\;& sech^2\left(\frac{x}{4}\right)+\left(\frac{1}{2}sech^2\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right)+\frac{1}{2}sech^4\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right)\right)t+\\ &+\frac{1}{16}\frac{1}{cosh^8\left(\frac{x}{4}\right)}\left(\begin{array}{c}\left(2sinh\left(\frac{x}{4}\right)cosh^5\left(\frac{x}{4}\right)-15sinh\left(\frac{x}{4}\right)cosh\left(\frac{x}{4}\right)\right)t\\ \left(-8cosh^2\left(\frac{x}{4}\right)+4cosh^6\left(\frac{x}{4}\right)+10cosh^4\left(\frac{x}{4}\right)-14\right)\frac{t^2}{2}\end{array}\right)\\ &+\frac{1}{128}\frac{1}{cosh^{11}\left(\frac{x}{4}\right)}\left\{\begin{array}{c}\left(\begin{array}{c}4sinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right)+4sinh\left(\frac{x}{4}\right)cosh^8\left(\frac{x}{4}\right)-300sinh\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right)\\+420sinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right)\end{array}\right)t\\ \left(-422cosh\left(\frac{x}{4}\right)+16cosh^9\left(\frac{x}{4}\right)+80cosh^7\left(\frac{x}{4}\right)-204cosh^3\left(\frac{x}{4}\right)+639cosh\left(\frac{x}{4}\right)\right)\frac{t^2}{2}\\ \left(\begin{array}{c}8sinh\left(\frac{x}{4}\right)cosh^8\left(\frac{x}{4}\right)+56sinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right)-6sinh^6\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right)\\-160sinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right)-90sinh\left(\frac{x}{4}\right)\end{array}\right)\frac{2t^3}{6}\end{array}\right\}\end{aligned}$$

4.2. Using Elzaki Projected Differential Transform Method

$$u_t(x,t)+u_x(x,t)+u(x,t)u_x(x,t)-u(x,t)u_{xxt}(x,t)=0 \tag{50}$$

With $u(x,0)=sech^2\left(\frac{x}{4}\right)$

Taking the Elzaki Transform of Eq. (42), it gives

$$E\left[u_t(x,t)\right]=E\Big[u_{xxt}(x,t)-u_x(x,t)-u(x,t)u_x(x,t)\Big] \tag{51}$$

Now, applying the differentiation property of Elzaki transform to gets

$$E\left[u(x,t)\right]=v^2u(x,0)+vE\Big[u_{xxt}(x,t)-u_x(x,t)-u(x,t)u_x(x,t)\Big] \tag{52}$$

Put $u(x,0)=sech^2\left(\frac{x}{4}\right)$ in Eq. (52), we have

$$E\left[u(x,t)\right]=v^2sech^2\left(\frac{x}{4}\right)+vE\Big[u_{xxt}(x,t)-u_x(x,t)-u(x,t)u_x(x,t)\Big] \tag{53}$$

Now, taking the inverse Elzaki transform of (53) we get:

$$E^{-1}\left\{E\left[u(x,t)\right]\right\}=E^{-1}\left\{v^2sech^2\left(\frac{x}{4}\right)\right\}+E^{-1}\left\{vE\Big[u_{xxt}(x,t)-u_x(x,t)-u(x,t)u_x(x,t)\Big]\right\} \tag{54}$$

Then,

$$u(x,t)=sech^2\left(\frac{x}{4}\right)+E^{-1}\left\{vE\Big[u_{xxt}(x,t)-u_x(x,t)-u(x,t)u_x(x,t)\Big]\right\} \tag{55}$$

And we have the recursive relation to be

$$\sum_{k=0}^{\infty}u(x,k+1)=E^{-1}\left\{vE\Big[u_{xxt}(x,t)-u_x(x,t)-u(x,t)u_x(x,t)\Big]\right\} \tag{56}$$

By applying the PDTM on (56), we have

$$u(x,k+1)=E^{-1}\left\{vE\Big[A_k-B_k-C_k\Big]\right\} \tag{57}$$

Note, $A_k = [u(x,k)]_{xxt}$, $B_k = [u(x,k)]_x$ and $C_k = \sum_{n=0}^{k} u(x,n)(u(x,k-n))_x$ are the projected differential transform of $u_{xxt}(x,t), u_x(x,t)$ and $u(x,t)u_x(x,t)$ respectively.

Now, taking $k=0$ and following the formal procedure in Step 1, we have

$$u(x,1) = E^{-1}\left\{vE\Big[A_0 - B_0 - C_0\Big]\right\} \tag{58}$$

$A_0 = [u(x,0)]_{xxt} = 0;$

$B_0 = [u(x,0)]_x = -\frac{1}{2}sech^2\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right)$

$C_0 = \sum_{n=0}^{k=0} u(x,n)(u(x,k-n))_x = u(x,0)(u(x,0))_x = -\frac{1}{2}sech^4\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right)$

Now, by substituting the projected differential and taking the Elzaki transform as described above we have,

$$u(x,1) = \left(\frac{1}{2}sech^2\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right) + \frac{1}{2}sech^4\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right)\right).t \tag{59}$$

For $k=1$

$$u(x,2) = E^{-1}\left\{vE\Big[A_1 - B_1 - C_1\Big]\right\} \tag{60}$$

$$A_1 = [u(x,1)]_{xxt} = \frac{1}{16}sinh\left(\frac{x}{4}\right)\frac{\left(2cosh^4\left(\frac{x}{4}\right) + 2cosh^2\left(\frac{x}{4}\right) - 15\right)}{cosh^7\left(\frac{x}{4}\right)}$$

$$B_1 = [u(x,1)]_x = -\frac{1}{8}\frac{t\left(2cosh^4\left(\frac{x}{4}\right) + cosh^2\left(\frac{x}{4}\right) - 5\right)}{cosh^6\left(\frac{x}{4}\right)}$$

$$\begin{aligned} C_1 &= \sum_{n=0}^{k=1} u(x,n)(u(x,k-n))_x = u(x,0)(u(x,1))_x + u(x,1)(u(x,0))_x \\ &= -\frac{1}{8}\frac{t\left(4cosh^4\left(\frac{x}{4}\right) + cosh^2\left(\frac{x}{4}\right) - 7\right)}{cosh^8\left(\frac{x}{4}\right)} \end{aligned}$$

Now, by substituting the projected differential and taking the Elzaki transform as described above we have,

$$u(x,2) = \frac{1}{16}\frac{1}{cosh^8\left(\frac{x}{4}\right)}\left(\begin{array}{c} \left(2sinh\left(\frac{x}{4}\right)cosh^5\left(\frac{x}{4}\right) + 2sinh\left(\frac{x}{4}\right)cosh^3\left(\frac{x}{4}\right) - 15sinh\left(\frac{x}{4}\right)cosh\left(\frac{x}{4}\right)\right) \\ + \left(-8cosh^2\left(\frac{x}{4}\right) + 4cosh^6\left(\frac{x}{4}\right) + 10cosh^4\left(\frac{x}{4}\right) - 14\right)\frac{t^2}{2} \end{array}\right)$$

For $k=2$

$$u(x,3) = E^{-1}\left\{vE\Big[A_2 - B_2 - C_2\Big]\right\} \tag{61}$$

$$\begin{aligned} A_2 &= [u(x,2)]_{xxt} \\ &= -\frac{1}{32}\frac{1}{cosh^{10}\left(\frac{x}{4}\right)} \\ &\left\{\begin{array}{c} \left(sinh\left(\frac{x}{4}\right)cosh^7\left(\frac{x}{4}\right) + sinh\left(\frac{x}{4}\right)cosh^5\left(\frac{x}{4}\right) + 105sinh\left(\frac{x}{4}\right)cosh\left(\frac{x}{4}\right) - 75sinh\left(\frac{x}{4}\right)cosh^3\left(\frac{x}{4}\right)\right) \\ + \left(17cosh^6\left(\frac{x}{4}\right) - 61cosh^4\left(\frac{x}{4}\right) - 70cosh^2\left(\frac{x}{4}\right) + 2cosh^8\left(\frac{x}{4}\right) + 126\right)t \end{array}\right\} \end{aligned}$$

$$\begin{aligned} B_2 &= [u(x,2)]_x \\ &= -\frac{1}{64}\frac{1}{cosh^9\left(\frac{x}{4}\right)}\left\{\begin{array}{c} \left(4cosh^7\left(\frac{x}{4}\right) + 2cosh^5\left(\frac{x}{4}\right) - 100cosh^3\left(\frac{x}{4}\right) + 105cosh\left(\frac{x}{4}\right)\right)t \\ + \left(4sinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right) + 20sinh\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right) - 24sinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right) - 56sinh\left(\frac{x}{4}\right)\right)t^2 \end{array}\right\} \end{aligned}$$

$$\begin{aligned} C_2 &= \sum_{n=0}^{k=1} u(x,n)(u(x,k-n))_x = u(x,0)(u(x,2))_x + u(x,1)(u(x,1))_x + u(x,2)(u(x,0))_x \\ &= -\frac{1}{64}\frac{1}{cosh^{11}\left(\frac{x}{4}\right)}\left\{\begin{array}{c} \left(8cosh^7\left(\frac{x}{4}\right) - 134cosh^3\left(\frac{x}{4}\right) + 2cosh^5\left(\frac{x}{4}\right) + 135cosh\left(\frac{x}{4}\right)\right)t \\ \left(-90sinh\left(\frac{x}{4}\right) + 16sinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right) + 42sinh\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right) - 48sinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right)\right)t^2 \end{array}\right\} \end{aligned}$$

Now, by substituting the projected differential and taking the Elzaki transform as described above we have,

$$u(x,3)=\frac{1}{64}\frac{1}{cosh^{11}\left(\frac{x}{4}\right)}\left\{\begin{array}{c}\left(\begin{array}{c}2cosh^8\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)+2sinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right)-150sinh\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right)\\+210sinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right)\Big)t\end{array}\right)t\\ \left(387cosh\left(\frac{x}{4}\right)-169cosh^3\left(\frac{x}{4}\right)-220cosh^5\left(\frac{x}{4}\right)+44cosh^7\left(\frac{x}{4}\right)+8cosh^9\left(\frac{x}{4}\right)\right)\frac{t^2}{2}\\ \left(\begin{array}{c}18cosh^4\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)-104cosh^2\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)+36cosh^6\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)\\-90sinh\left(\frac{x}{4}\right)+4cosh^8\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)\end{array}\right)\frac{t^3}{3}\end{array}\right\}$$

And so on, then the solution of the BBM Eq. (50) using EPDTM with initial condition $u(x,0)=sech^2\left(\frac{x}{4}\right)$ is given as

$$u(x,t)=sech^2\left(\frac{x}{4}\right)+\left(\frac{1}{2}sech^2\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right)+\frac{1}{2}sech^4\left(\frac{x}{4}\right)tanh\left(\frac{x}{4}\right)\right).t$$

$$+\frac{1}{16}\frac{1}{cosh^8\left(\frac{x}{4}\right)}\left\{\begin{array}{c}\left(2sinh\left(\frac{x}{4}\right)cosh^5\left(\frac{x}{4}\right)+2sinh\left(\frac{x}{4}\right)cosh^3\left(\frac{x}{4}\right)-15sinh\left(\frac{x}{4}\right)cosh\left(\frac{x}{4}\right)\right)t\\+\left(-8cosh^2\left(\frac{x}{4}\right)+4cosh^6\left(\frac{x}{4}\right)+10cosh^4\left(\frac{x}{4}\right)-14\right)\frac{t^2}{2}\end{array}\right\}$$

$$+\frac{1}{64}\frac{1}{cosh^{11}\left(\frac{x}{4}\right)}\left\{\begin{array}{c}\left(\begin{array}{c}2cosh^8\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)+2sinh\left(\frac{x}{4}\right)cosh^6\left(\frac{x}{4}\right)-150sinh\left(\frac{x}{4}\right)cosh^4\left(\frac{x}{4}\right)\\+210sinh\left(\frac{x}{4}\right)cosh^2\left(\frac{x}{4}\right)\Big)t\end{array}\right)t\\ \left(387cosh\left(\frac{x}{4}\right)-169cosh^3\left(\frac{x}{4}\right)-220cosh^5\left(\frac{x}{4}\right)+44cosh^7\left(\frac{x}{4}\right)+8cosh^9\left(\frac{x}{4}\right)\right)\frac{t^2}{2}\qquad+\dots\\ \left(\begin{array}{c}18cosh^4\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)-104cosh^2\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)+36cosh^6\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)-90sinh\left(\frac{x}{4}\right)\\+4cosh^8\left(\frac{x}{4}\right)sinh\left(\frac{x}{4}\right)\Big)\frac{t^3}{3}\end{array}\right)\end{array}\right\}\dots$$

5. Results and discussion

In this section, we checked the results of the two methods (LADM and EPDTM) for the two studies (study 1 and study 2) in order to validate the efficacy, convergence, and authenticity of the two methods and draw a valid conclusion on which method is best.

Study 1 has the following exact solution $u(x,t)=\dfrac{x-t}{1+t}$ and Study 2 has the following exact solution $u(x,t)=sech^2\left(\frac{x}{4}-\frac{t}{3}\right)$

5.1. Study 1: Solution of BBM equation (1) with initial condition given by $u(x,0)=x$

We prepare Table 1 which we compare the exact results with LADM and EPDTM results in order to validate the efficacy and draw a valid conclusion on which method is the best.

5.2. Study 2: Solution of BBM equation (1) with initial condition given by $u(x,0)=sech^2\left(\frac{x}{4}\right)$

We prepare Table 2 which we compare the exact results with LADM and EPDTM results in order to validate the efficacy and draw a valid conclusion on which method is the best.

5.3. Plots of the solution and convergence of the exact, LADM, and EPDTM dolutions

Figs. 1 to 6 show the plots of the solution for the study 1 and 2. Also, we presented the convergence analysis plots in Figs. 7 to 10.

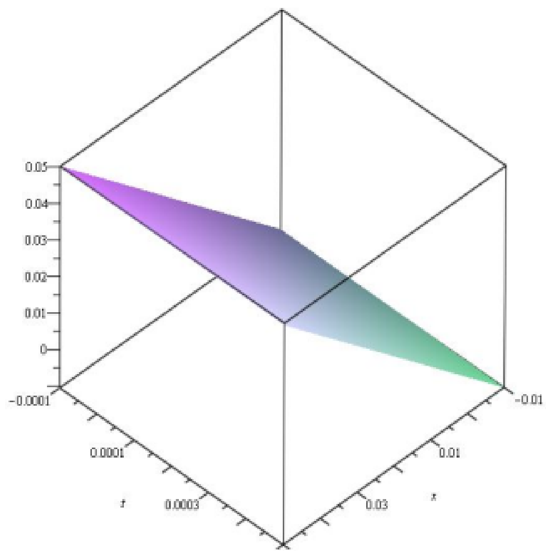

Figure 1: Exact solution plot for BBM Equation (Study 1).

Table 1: The exact result, LADM result and EPDTM result of BBM Equation for Study 1 conditions.

Study 1	t	Exact	LADM	EPDTM	Error=|Exact-LADM|	Error=|Exact-EPDTM|
x=0.01	0.0001	0.009899010099	0.009899010099	0.009899010099	0.000000000000	0.000000000000
	0.0002	0.009798040392	0.009798040395	0.009798040392	0.000000000003	0.000000000000
	0.0003	0.009697090873	0.009697090882	0.009697090873	0.000000000009	0.000000000000
	0.0004	0.009596161535	0.009596161557	0.009596161535	0.000000000022	0.000000000000
	0.0005	0.009495252374	0.009495252416	0.009495252374	0.000000000042	0.000000000000
x=0.02	0.0001	0.019898010200	0.019898010200	0.019898010200	0.000000000000	0.000000000000
	0.0002	0.019796040790	0.019796040790	0.019796040790	0.000000000000	0.000000000000
	0.0003	0.019694091770	0.019694091780	0.019694091770	0.000000000010	0.000000000000
	0.0004	0.019592163130	0.019592163160	0.019592163140	0.000000000030	0.000000000010
	0.0005	0.019490254870	0.019490254920	0.019490254880	0.000000000050	0.000000000010
x=0.03	0.0001	0.029897010300	0.029897010300	0.029897010300	0.000000000000	0.000000000000
	0.0002	0.029794041190	0.029794041190	0.029794041190	0.000000000000	0.000000000000
	0.0003	0.029691092670	0.029691092680	0.029691092670	0.000000000010	0.000000000000
	0.0004	0.029588164730	0.029588164760	0.029588164740	0.000000000030	0.000000000010
	0.0005	0.029485257370	0.029485257420	0.029485257380	0.000000000050	0.000000000010
x=0.04	0.0001	0.039896010400	0.039896010400	0.039896010400	0.000000000000	0.000000000000
	0.0002	0.039792041590	0.039792041590	0.039792041590	0.000000000000	0.000000000000
	0.0003	0.039688093570	0.039688093580	0.039688093570	0.000000000010	0.000000000000
	0.0004	0.039584166330	0.039584166360	0.039584166340	0.000000000030	0.000000000010
	0.0005	0.039480259870	0.039480259920	0.039480259880	0.000000000050	0.000000000010
x=0.05	0.0001	0.049895010500	0.049895010500	0.049895010500	0.000000000000	0.000000000000
	0.0002	0.049790041990	0.049790041990	0.049790041990	0.000000000000	0.000000000000
	0.0003	0.049685094470	0.049685094480	0.049685094470	0.000000000010	0.000000000000
	0.0004	0.049580167930	0.049580167960	0.049580167940	0.000000000030	0.000000000010
	0.0005	0.049475262370	0.049475262420	0.049475262370	0.000000000050	0.000000000000

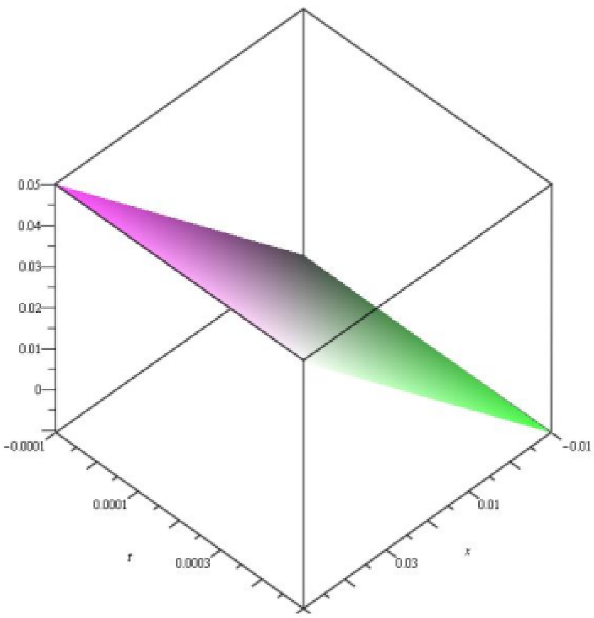

Figure 2: LADM solution plot for BBM Equation (Study 1).

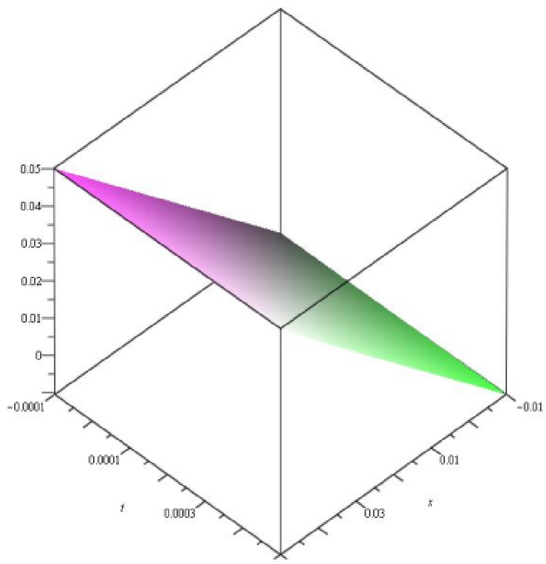

Figure 3: EPDTM solution plot for BBM Equation (Study 1).

Table 2: The exact result, LADM result and EPDTM result of BBM Equation for Study 2 conditions.

Study 2	t	Exact	LADM	EPDTM	Error=|Exact-LADM|	Error=|Exact-EPDTM|
x=0.01	0.0001	0.99999392	0.99999357	0.99999408	0.00000034	0.00000016
	0.0002	0.99999408	0.99999338	0.99999441	0.0000007	0.00000033
	0.0003	0.99999424	0.99999318	0.99999475	0.00000106	0.00000051
	0.0004	0.9999944	0.99999297	0.99999508	0.00000143	0.00000069
	0.0005	0.99999456	0.99999274	0.99999543	0.00000182	0.00000087
x=0.02	0.0001	0.99997533	0.99997465	0.99997566	0.00000068	0.00000033
	0.0002	0.99997566	0.99997429	0.99997632	0.00000137	0.00000066
	0.0003	0.99997599	0.99997392	0.99997698	0.00000279	0.00000133
	0.0004	0.99997632	0.99997353	0.99997765	0.00000279	0.00000133
	0.0005	0.99997664	0.99997313	0.99997832	0.00000351	0.00000168
x=0.03	0.0001	0.99994425	0.99994323	0.99994474	0.00000102	0.00000049
	0.0002	0.99994475	0.9999427	0.99994573	0.00000205	0.00000098
	0.0003	0.99994524	0.99994215	0.99994672	0.00000309	0.00000148
	0.0004	0.99994573	0.9999416	0.99994771	0.00000414	0.00000198
	0.0005	0.99994622	0.99994103	0.99994871	0.0000052	0.00000248
x=0.04	0.0001	0.99990067	0.99989931	0.99990132	0.00000136	0.00000065
	0.0002	0.99990134	0.99989861	0.99990264	0.00000273	0.0000013
	0.0003	0.999902	0.99989789	0.99990396	0.0000041	0.00000196
	0.0004	0.99990266	0.99989716	0.99990528	0.00000549	0.00000262
	0.0005	0.99990331	0.99989642	0.9999066	0.00000689	0.00000329
x=0.05	0.0001	0.9998446	0.9998429	0.99984541	0.0000017	0.00000081
	0.0002	0.99984543	0.99984203	0.99984705	0.0000034	0.00000162
	0.0003	0.99984626	0.99984114	0.9998487	0.00000512	0.00000244
	0.0004	0.99984708	0.99984024	0.99985035	0.00000684	0.00000327
	0.0005	0.9998479	0.99983933	0.999852	0.00000858	0.00000409

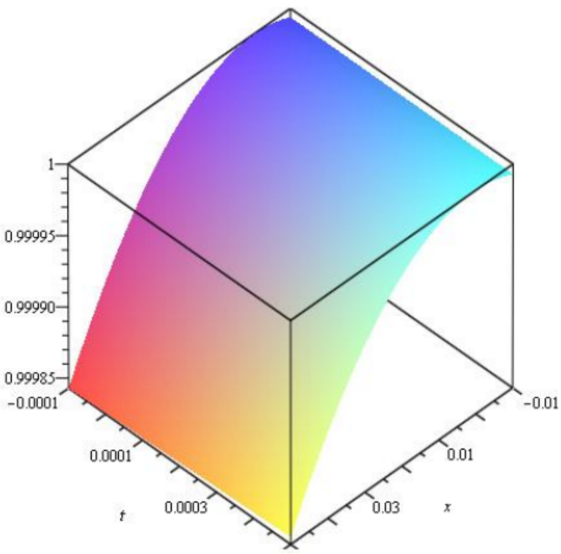

Figure 4: Exact solution plot for BBM Equation (Study 2).

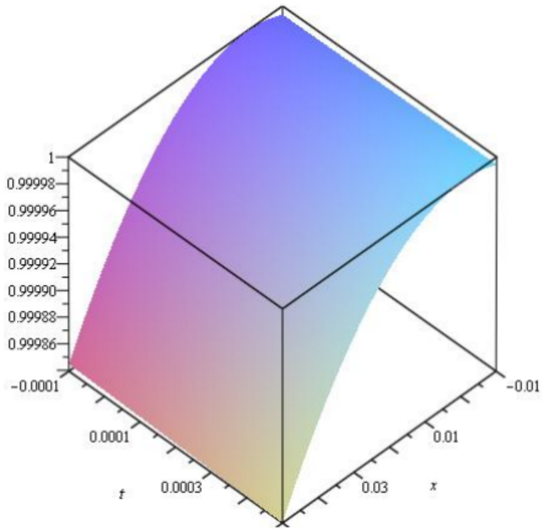

Figure 5: LADM solution plot for BBM Equation (Study 2).

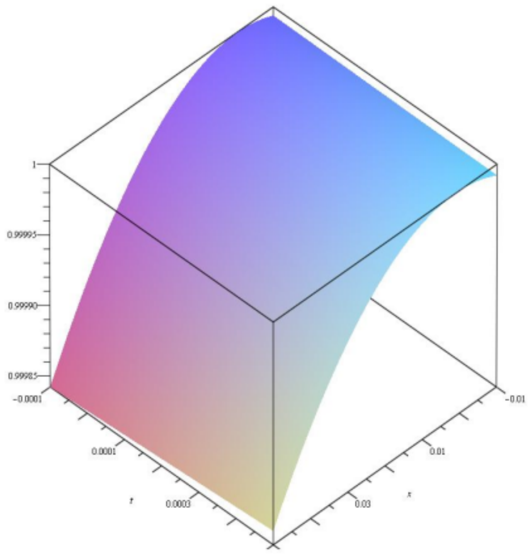

Figure 6: EPDTM solution plot for BBM Equation (Study 2).

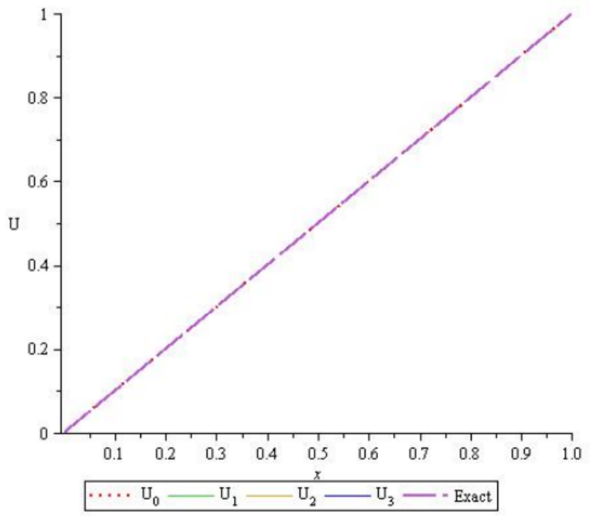

Figure 7: Convergent plot of the LADM solution plot for BBM Equation (Study 1).

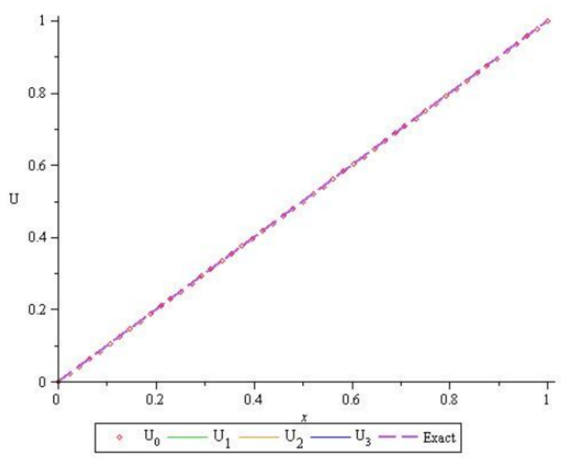

Figure 8: Convergent plot of the EPDTM solution plot for BBM Equation (Study 1).

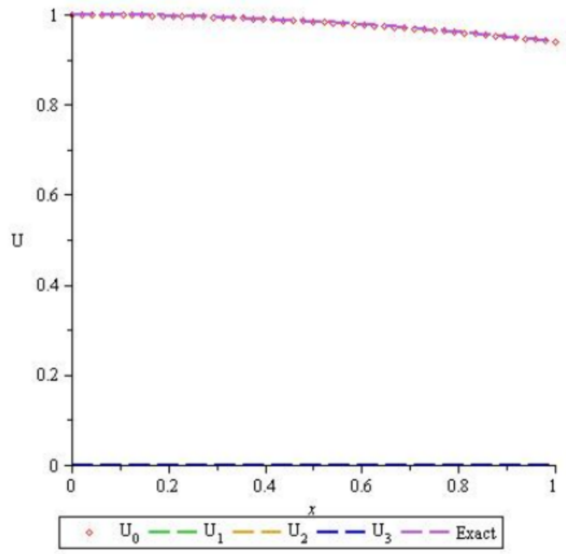

Figure 9: Convergent plot of the EPDTM solution plot for BBM Equation (Study 2).

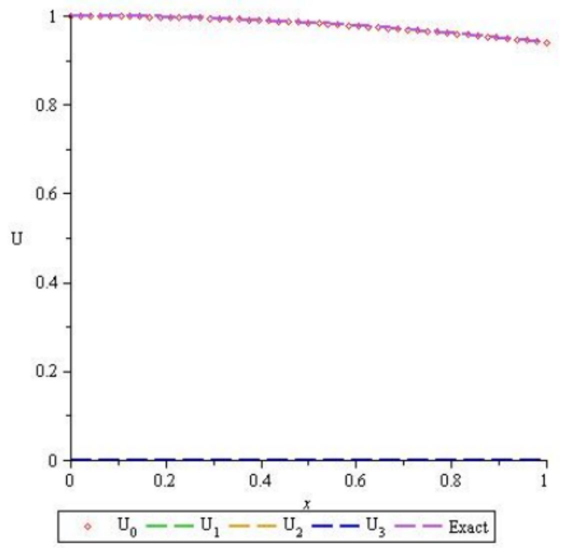

Figure 10: Convergent plot of the LADM solution plot for BBM Equation (Study 2).

5.4. Discussion

In this research, we compare and contrast two numerical methods, the Laplace Adomian Decomposition Method (LADM) and the Elzaki Projected Differential Transform Method (EPDTM), for solving the Benjamin-Bona-Mahony (BBM) equation, and this helps in evaluating their performance, assessing robustness, gaining insights into solution behavior, validating results, generalizing their applicability, and advancing the field.

Table 1 and Table 2 shows the results of BBM equations using LADM and EPDTM and we calculated the error of these two methods by comparing their results with the exact solution. And we see that EPDTM was able to approximate the true solution of BBM equation with a high level of precision, even in the presence of strong nonlinearities also seeing Figs. 1 to 10, EPDTM was able demonstrate high convergence i.e. the solutions approach the exact solution as the resolution increases. And this proved that EPDTM is consistent and reliable solutions that approximate the exact solution compare to LADM.

Our results shows that the Elzaki Projected Differential Transform Method has high efficiency, flexibility and robustness compare to Laplace Adomian Decomposition method. Also it was noted that EPDTM is a user-friendly, which allows clear and concise implementation procedures to solving of BBM equation without excessive difficulties, (see Eq. 9-12, 25-32 and 46-60).

6. Conclusion

The results obtained using the Elzaki Projected Differential Transform Method compared to the Laplace Adomian Decomposition Method show that the Elzaki Projected Differential Transform Method has high accuracy, convergence, stability, robustness, efficiency, flexibility, consistency, and user friendliness for solving nonlinear PDEs in their specific applications.

Elzaki Projected Differential Transform Method having a simple and accurate numerical method to solve the Benjamin-Bona-Mahony (BBM) equation is beneficial for several reasons:

1. **Efficiency:** EPDTM allows for efficient computation of solutions to the BBM equation. The BBM equation, being a nonlinear partial differential equation, can be computationally demanding to solve. A simple numerical method reduces the computational complexity and time required for obtaining solutions, enabling faster simulations and analysis.

2. **Accessibility:** EPDTM is more accessible to researchers, engineers, and students who may not have expertise in advanced numerical techniques. It allows a wider range of users to apply the method without requiring extensive knowledge of complex algorithms or specialized software. This accessibility promotes the use of the BBM equation and facilitates its application in various fields.

3. **Implementation and code development:** EPDTM simplifies the implementation process and the development of computational codes for solving the BBM equation. It reduces the potential for errors and improves code maintainability. Researchers can focus more on studying the physical and mathematical aspects of the BBM equation rather than struggling with complex implementation details.

4. **Accuracy:** The accuracy of the EPDTM is essential for obtaining reliable solutions to the BBM equation. EPDTM achieve a high level of accuracy ensures that the computed solutions closely match the true behavior of the wave dynamics described by the BBM equation. Accurate solutions are crucial for making reliable predictions, analyzing wave behavior, and designing coastal structures.

5. **Robustness:** EPDTM is robust and stable ensures that the computed solutions remain accurate and reliable even in the presence of challenging conditions, such as steep gradients, wave breaking, or shock formation. Robustness is particularly important for the BBM equation, as it models wave propagation and wave breaking, which can involve complex and dynamic phenomena.

6. **Validation and verification:** EPDTM enables the validation and verification of the solutions obtained for the BBM equation. Comparing the numerical results with known analytical solutions, if available, or experimental data helps validate the accuracy of the method. Verification ensures that EPDTM is correctly implemented and provides reliable solutions for the BBM equation.

Having EPDTM as a simple and accurate numerical method for solving the BBM equation facilitates its practical applications in various fields, including coastal engineering, wave modeling, and wave stability analysis. It enhances accessibility, computational efficiency, accuracy, robustness, and validation of the solutions, supporting the understanding and analysis of wave phenomena described by the BBM equation.

Overall, we recommend the Elzaki Projected Differential Transform Method as an appropriate numerical method that is essential for obtaining accurate, reliable, and efficient solutions to complex problems. It ensures that the numerical analysis aligns with the specific requirements and constraints of the problem, leading to trustworthy results and valuable insights for scientific, engineering, or real-world applications.

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