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APPROXIMATION OF THE CONJUGATE OF A FUNCTION BELONGING TO THE W (L^P, ξ (t)) CLASS BY (N, P_n) (E, 1) MEANS OF THE CONJUGATE SERIES OF THE FOURIER SERIES

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ABSTRACT
In this paper, a theorem concerning the degree of approximation of the conjugate of a function belonging to W (L^p, ξ (t)) class by (N, p_n) (E, 1) means of its conjugate series of a Fourier series has been proved.
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Key words and phrases: Degree of approximation, W (L^p, ξ(t)) class, (N, p_n)(E, 1) summability, Fourier series, conjugate series of a Fourier series.

INTRODUCTION

Bernstin (1912), used (C,1) means to obtain the degree of approximation $E_n(f) = O\left(\frac{\log n}{n}\right)$ by Lip1 class. Jackson (1930) determined $E_n(f) = O\left(\frac{1}{n}\right)$ by using (C, δ) method in Lip α class, for 0<α<1. Qureshi (1981), first time obtained the degree of approximation of the function $\tilde{f}(x)$ i.e., $E_n(\tilde{f}) = O\left(\frac{1}{P_n}\sum_{k=1} \frac{p_k}{k^{\alpha+1}}\right)$, $0 < \alpha < 1$, by Nörlund means, where $\tilde{f}(x)$ is the conjugate of 2π-periodic function f∈Lip α. Generalizing the result of Qureshi (1981), many interesting results have been proved by various investigators like Qureshi (1982), Lal (2000), Lal and Nigam (2001), Rhoades (2002), Mittal *et. al.* (2005) for functions of various classes Lip α, Lip (α, p), Lip (ξ(t), p) and W(L^p, ξ (t)) by using various summability methods.

Let f be 2π-periodic, integrable over (-π,π) in the sense of Lebesgue, then its Fourier series is given by

$$f(t) \approx \frac{1}{2}a_o + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \tfrac{1}{2}a_o + \sum_{n=1}^{\infty} A_n(t) \qquad (1)$$

with partial sum $S_n(x)$.
The conjugate series of the Fourier series (1) given by

$$\sum_{n=1}^{\infty} (a_n \sin nt - b_n \cos nt) = -\sum_{n=1}^{\infty} B_n(t) \tag{2}$$

with partial sum $\tilde{S}_n(x)$.

We define

$$t_n^{NE} = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} \frac{1}{2^k}\sum_{r=0}^{k} \binom{k}{r} S_r ,$$

where t_n^{NE} is $(N, p_n)(C,1)$ means of the sequence $\{S_n\}$, if $t_n^{NE} \to S$ as $n \to \infty$, then sequence $\{S_n\}$ is summable by $(N, p_n)(C,1)$ method to S.

The L^p norm is defined by

$$\| f \|_p = \left(\int_0^{2\pi} | f(x) |^p dx \right)^{\frac{1}{p}}, \ p \geq 1,$$

and the degree of approximation $E_n(f)$ under norm $\| \ \|_p$ is given by (Zygmund, 1959)

$$E_n(f) = \min_{T_n} \| T_n - f \|_p,$$

where T_n is a trigonometric polynomial of degree n.

A function $f \in Lip\ \alpha$ if

$$| f(x+t) - f(x) | = O\left(|t|^{\alpha} \right), \ \text{for } 0 < \alpha \leq 1.$$

Also, $f \in Lip(\alpha, p)$, for $0 \leq x \leq 2\pi$, if

$$\left(\int_0^{2\pi} | f(x+t) - f(x) |^p dx \right)^{\frac{1}{p}} = O\left(|t|^{\alpha} \right), \ 0 < \alpha \leq 1, \ p \geq 1.$$

Given a positive increasing function $\xi(t)$, $p \geq 1$,
$f \in Lip(\xi(t), p)$ if

$$\left(\int_0^{2\pi} \left| \left(f(x+t) - f(x) \right) \right|^p dx \right)^{\frac{1}{p}} = O(\xi(t)), \quad \text{and}$$

$f \in W(L^p, \xi(t))$ if $\left(\int_0^{2\pi} \left| \left(f(x+t) - f(x) \right) \sin^{\beta} x \right|^p dx \right)^{\frac{1}{p}} = O(\xi(t))$, $(\beta \geq 0)$ (Rhoades, 2002).

It is observed that

$$W(L^p, \xi(t)) \xrightarrow{\beta=0} Lip(\xi(t), p) \xrightarrow{\xi(t)=t^{\alpha}} Lip(\alpha, p) \xrightarrow{p\to\infty} Lip\alpha.$$

We write

$$\psi(t) = f(x+t) - f(x-t).$$

$$\tilde{K}(n,t) = \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_k \frac{\cos (n-k+1)\frac{t}{2} \cos^{n-k}\left(\frac{t}{2}\right)}{\sin \frac{t}{2}} \tag{3}$$

$\tau = \left[\frac{1}{t}\right]$, where, τ denotes the greatest integer not greater than $\frac{1}{t}$.

THEOREM

The purpose of this paper is to obtain the approximation of $\tilde{f}(x)$, the conjugate of a function

$f \in W(L^p, \xi(t))$ class, by (N, p_n) (E, 1) means of conjugate series of a Fourier series. In fact, we prove following theorem:

The degree of approximation of function $\tilde{f}(x)$, conjugate to 2π-periodic, Lebesgue integrable

in $(-\pi, \pi)$ function f(x) belonging to class $W(L^p, \xi(t))$, $p \geq 1$, by using $\tilde{t}_n^{NE}(x)$ on

its conjugate Fourier series (2), is given by

$$\left\| \tilde{t}_n^{NE} - \tilde{f} \right\| = O\left(n^{\beta + \frac{1}{p}} \xi\left(\tfrac{1}{n}\right) \right), \tag{4}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left\{ \int_0^{\frac{1}{n}} \left(\frac{\psi(t)| (t) |}{\xi(t)} \right)^p \sin^{\beta p} t dt \right\}^{\frac{1}{p}} = O\left(\frac{1}{n}\right) \tag{5}$$

and
$$\left\{ \int_{\frac{1}{n}}^{\pi} \left(\frac{t^{-\delta} |\psi(t)|}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} = O\left(n^{\delta}\right), \tag{6}$$

uniformly in x, where δ is an arbitrary number with $(1-\delta) - \frac{1}{q} > 0$, q is the conjugate index of

p, $\tilde{t}_n^{NE} = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \frac{1}{2^k} \sum_{r=0}^{k} \binom{k}{r} \tilde{S}_r$

is the (N, p_n) (E, 1) means and $\tilde{f}(x) = -\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot \frac{t}{2} dt$, exists in the sense of Lebesgue.

LEMMAS

Lemma 1: For $0 < t < \frac{1}{n}$ and fact that $\frac{1}{\sin t} \leq \frac{\pi}{2t}$ for $0 < t \leq \frac{\pi}{2}$,

$$\tilde{K}(n,t) = O\left(\frac{1}{t}\right). \tag{7}$$

Proof: $$\left|\tilde{K}(n,t)\right| \leq \frac{1}{2\pi P_n} \sum_{k=0}^{n} p_k \left| \frac{\cos\,(n-k+1)\frac{t}{2} \cos^{n-k}\left(\frac{t}{2}\right)}{\sin \frac{t}{2}} \right|$$

$$\leq \frac{1}{2t\,P_n} \sum_{k=0}^{n} p_k \left|\cos\,(n-k+1)\tfrac{t}{2}\cos^{n-k}\left(\tfrac{t}{2}\right)\right|$$
$$= O\left(\frac{1}{t}\right).$$

Lemma 2: If $\{p_n\}$ is non-negative and non-increasing sequence, then for $0 \leq a \leq b \leq \infty$, $0 \leq t \leq \pi$ and for any n,

$$\left|\sum_{k=a}^{b} p_k e^{i(n-k)t}\right| = O(P_\tau),\ \text{where}\ \ \tau = [\tfrac{1}{t}] \quad \text{(McFadden, 1942).} \tag{8}$$

Lemma 3: For $\frac{1}{n} < t < \pi$,

$$\tilde{K}(n,t) = O\left(\frac{P_\tau}{tP_n}\right). \tag{9}$$

Proof:

$$\left|\tilde{K}(n,t)\right| \leq \frac{1}{2t\,P_n}\left|\sum_{k=0}^{n} p_k \cos\,(n-k+1)\tfrac{t}{2}\cos^{n-k}\left(\tfrac{t}{2}\right)\right|$$
$$\leq \frac{1}{2t\,P_n}\left|\sum_{k=0}^{n} \text{Real part of } p_k\ e^{i(n-k+1)\frac{t}{2}}\right|$$
$$\leq \frac{1}{2t\,P_n}\left|\sum_{k=0}^{n} \text{Real part of } p_k\ e^{i(n-k)t}\right|$$
$$\leq \frac{1}{2t\,P_n}\left|\sum_{k=0}^{n} p_k\ e^{i(n-k)t}\right|$$
$$= O\left(\frac{P_\tau}{tP_n}\right) \text{ by using Lemma 2.}$$

PROOF OF THE THEOREM

The n th partial sum $\tilde{S}_n(x)$ of the series (2) is given by

$$\tilde{S}_n(x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{\psi(t)\ \cos(n+\frac{1}{2})t}{\sin\frac{t}{2}}\,dt \text{ . So that,}$$

$\tilde{t}_n^{NE}(x)$ transform of $\tilde{S}_n(x)$ is

$$\tilde{t}_n^{NE}(x) - \tilde{f}(x) = \int_0^{\pi} \psi(t)\frac{1}{2\pi P_n}\sum_{k=0}^{n} p_k\ \frac{\cos\,(n-k+1)\frac{t}{2}\cos^{n-k}\left(\frac{t}{2}\right)}{\sin\frac{1}{2}\ t}\,dt$$
$$= \int_0^{\pi} \psi(t)\ \tilde{K}(n,t)\,dt$$

$$= \int_0^{\frac{1}{n}} \psi(t)\, \tilde{K}(n,t)\, dt + \int_{\frac{1}{n}}^{\pi} \psi(t)\, \tilde{K}(n,t)\, dt$$

$$= I_1 + I_2 \text{, (say).} \tag{10}$$

Applying Hölder's inequality, condition (5), second mean value theorem for integral and the fact that $\psi(t) \in W(L^p, \xi(t))$, we have

$$|I_1| = \int_0^{\frac{1}{n}} |\psi(t)| \left|\tilde{K}(n,t)\right| dt$$

$$\le \left[\int_0^{\frac{1}{n}} \left(\frac{t\,|\,\psi(t)\,|\sin^\beta t}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} \left[\int_0^{\frac{1}{n}} \left(\frac{\xi(t)\,\left|\tilde{K}(n,t)\right|}{t\sin^\beta t}\right)^q dt\right]^{\frac{1}{q}}$$

$$= O\left(\frac{1}{n}\right) O\left[\left\{\int_\varepsilon^{\frac{1}{n}} \left(\frac{\xi(t)}{t^{2+\beta}}\right)^q dt\right\}^{\frac{1}{q}}\right]$$

$$= O\left(\frac{1}{n}\xi\left(\frac{1}{n}\right)\right) O\left[\left\{\int_\varepsilon^{\frac{1}{n}} t^{-(2+\beta)q}\, dt\right\}^{\frac{1}{q}}\right]$$

$$= O\left(\frac{1}{n}\xi\left(\frac{1}{n}\right)\right) O\left[\left\{\int_\varepsilon^{\frac{1}{n}} t^{-(2+\beta)q}\, dt\right\}^{\frac{1}{q}}\right]$$

$$= O\left(\frac{1}{n}\xi\left(\frac{1}{n}\right)\right) O\left(n^{2+\beta-\frac{1}{q}}\right)$$

$$= O\left(n^{\beta+\frac{1}{p}}\, \xi\left(\frac{1}{n}\right)\right). \tag{11}$$

Similarly as above, we have

$$|\,I_2\,| = \left[\int_{\frac{1}{n}}^{\pi} \left|\frac{t^{-\delta}\,\psi(t)}{\xi(t)}\sin^\beta t\right|^p dt\right]^{\frac{1}{p}} \left[\int_{\frac{1}{n}}^{\pi} \left|\frac{\xi(t)\,\tilde{K}(n,t)}{t^{-\delta}\sin^\beta t}\right|^q dt\right]^{\frac{1}{q}}$$

$$= O\left(n^\delta\right) O\left[\left\{\int_{\frac{1}{n}}^{\pi} \left(\frac{\xi(t)\,Q_\tau}{Q_n\, t^{(1-\delta+\beta)}}\right)^q dt\right\}^{\frac{1}{q}}\right]$$

$$= O\left(n^\delta\, \xi\left(\frac{1}{n}\right)\right) O\left[\left\{\int_{\frac{1}{n}}^{\pi} t^{q(\delta-\beta-1)} dt\right\}^{\frac{1}{q}}\right]$$

$$= O\left(n^{\delta}\,\xi\left(\frac{1}{n}\right)\right)\; O\left[\left\{\left(\frac{t^{q(\delta-\beta-1)+1}}{q(\delta-\beta-1)+1}\right)_{\frac{1}{n}}^{\pi}\right\}^{\frac{1}{q}}\right]$$

$$= O\left(n^{\beta+1-\frac{1}{q}}\,\xi\left(\frac{1}{n}\right)\right)$$

$$= O\left(n^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n}\right)\right). \qquad (12)$$

By using (10), (11) & (12), we get,

$$\left|\tilde{t}_n^{NE}(x) - \tilde{f}(x)\right| = O\left(n^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n}\right)\right)$$

that is,
$$\left\|\tilde{t}_n^{NE}(x) - \tilde{f}(x)\right\|_p = O\left(n^{\beta+\frac{1}{p}}\,\xi\left(\frac{1}{n}\right)\right).$$

APPLICATIONS

The following corollaries can be derived from the theorem.

Corollary 1. If $\beta = 0$ and $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$, then the $W(L^p, \xi(t))$ class reduces to Lip (α, p) class and the degree of approximation of a function $\tilde{f}(x)$, conjugate to 2π-periodic function $f \in$ Lip (α, p), is given by

$$\left\|\tilde{t}_n^{NE}(x) - \tilde{f}(x)\right\|_p = O\left(\frac{1}{n^{\alpha-\frac{1}{p}}}\right).$$

Corollary 2. If $p \to \infty$ in corollary 1, for $0 < \alpha < 1$, degree of approximation of a function $\tilde{f}(x)$, conjugate to 2π-periodic function $f \in$ Lip α, is given by

$$\left\|\tilde{t}_n^{NE}(x) - \tilde{f}(x)\right\|_\infty = \sup_{-\pi\le x\le\pi}\left|\tilde{t}_n^{NE}(x) - \tilde{f}(x)\right| = O\left(\frac{1}{n^{\alpha}}\right).$$

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