FOUNDATIONS OF GEOMETRY – I

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This article attempts to make a short but systematic presentation of the fundamentals of Euclidean geometry, non-Euclidean geometry of Lobachevsky and Riemann, Projective geometry and the geometrical aspects of special relativity theory.

1.1 Euclid's definitions, postulates and axioms

The origin of geometrical ideas can be traced back to distant past. The ancient cultures of Babylon and Egypt are usually credited with the development of geometrical ideas. Development of geometry by the Greek scholars began in the seventh century B.C. Many basic facts pertaining to geometry were developed in the sixth and fifth centuries B.C. and apparently the concept of the proof of a theorem had already been formulated during this time.

In the third century B.C. the Greeks had developed a deep knowledge of geometry, accumulated a large number of geometric facts, and also acquainted themselves with the methods of proofs of geometrical facts. Therefore it was natural that attempts were also made to collect all these results together and put them in a logical order. The task of describing the development of geometry was undertaken by many Greek authors but their works have not survived to the present day. All these works had been forgotten after the appearance of **Euclid's** famous **Elements.**

One of the most famous geometers of antiquity, Euclid lived in 300 B.C. His work Elements is divided into thirteen books, of which the fifth, seventh, eight, ninth and tenth are devoted to the theory of proportion and to arithmetic which the remaining parts of the book are geometry proper. The first book contains conditions for the equality of triangles, relationships between sides and angles of a triangle, the theory of parallel lines and conditions for triangles and polygons to be equal. The second book deals which the transformation of a polygon into squares of equal area. The third book is devoted to circles, while the fourth book discusses the inscribed and circumscribed polygons. The sixth book studies the similarity of polygons. Foundations of **solid geometry** are discussed in the last three books.

Thus the **Elements** contain essentially elementary geometry. All of the geometry already known in Euclid's time for example, the theory of conic section is not included there.

Euclid begins each of his books with definitions of these concepts that are required for the book in question. The first book opens with 23 definitions including the definitions of a point, a line, a surface, a plane and a plane angle. After these definitions Euclid lists postulates and axioms, that is, assertions whose truthfulness are accepted without proofs.

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The origin of some of these axioms is subject to some doubt. In some editions of the Elements the fourth and fifth postulates are treated as axioms; therefore the fifth postulate is sometimes referred to as the 11^{th} axiom. It is not, at all, clear by what principles statements were classified as either postulates or axioms.

The enumeration of definitions and axioms looked sufficient for the logical proof of all the theorems of geometry. Euclid himself used these as accurately as was possible in his time. What is more, for many centuries the rigorous proofs of Euclid were considered as model to be imitated. But from the viewpoint of modern mathematics the exposition of the Elements is unsatisfactory in many respects.

If we examine Euclid's definitions, we find these definitions formulated in terms of concepts which themselves need to be defined. For example, "extremity", "length" etc. are used without definition. The list of basic statements accepted by Euclid, without proof, can hardly form a basis for strictly logical development of Geometry. We frequently use phrases as, "two points lie on one side of the lie", two points line on opposite sides of the line", "a point lies within a polynomial", etc. Euclid's postulates do not give us any information to substantiate these concepts but such statements are used to prove theorems. We are forced to rely on some visual evidence. But in a strictly logical construction any proposition not included in the axioms must be proved no matter how evident it may seem to be.

Further, according to axiom 7 the equality of geometrical quantities and figures is determined with the aid of motion. Euclid does not define the concept of motion and also the properties of motion are not mentioned in any of the axioms. While proving the congruence of two triangles, one triangle is lifted in space and put over another these to show that these two triangles. Euclid took it granted that the shape of a geometrical figure is not distorted while being lifted in space and all of us followed him. There are other facts too which may look self-evident but they must be proved or they should be among the postulates and axioms.

Although the convincing nature of Euclid's logic corroborated with our habitual conception of space, nevertheless some of the defects of Euclid's Elements were noted by the scholars of antiquity. Noted among such scholars was Archimedes who brought Euclid's exposition regarding the theory of lengths, areas and volumes to near completion. Euclid only established ratios of Lengths, areas, and volumes showing, for example, that the area of a circle is proportional to the square of its radius and the volume of a sphere is proportional to the cube of the radius, Archimedes obtained mathematical expressions enabling the computation of these quantities.

Archimedes introduced five postulates. The first four postulates deal with the length of a curve, area of a surface and volume of a body. But these concepts, defined in other simpler geometrical definitions, can be set forth in a proper from to prove the propositions of Archimedes. Hence the postulates of Archimedes need not be treated as postulates. The last postulate, of course, is of great importance. This is stated as "if a and b are real numbers with $0 < a$, then there exists a natural number n such that $na > b$". This postulate is fundamental to the measurement of geometrical quantities.

After Archimedes, attempts to make the basic structure of geometry went on. But nothing substantial was added to what Euclid had done. Almost to the beginning of nineteenth century Euclid's proofs were considered sufficient. It was only at the end of nineteenth century that a strictly logical foundation of geometry was conceived and a complete system of axioms, sufficient to derive all theorems of geometry without anything to do with the concept of space, was found.

Many scientists dealing with the Elements tried to reduce the number of axioms. The attempt to introduce minimum number of axioms on which geometry could be developed continued. In this regard Euclid's fourth postulate dealing with the equality of right angles was found to be superfluous. Like many other propositions, this can be rigorously proved.

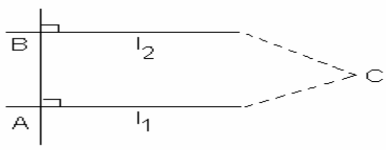
1.2 The fifth postulate

Euclid's fifth postulate drew attention of many geometers. Unlike other postulates, this one looked too complicated to be considered a postulate and so attempts were made to drop it out of postulate but prove it as a theorem. The investigations concerning this postulate continued till the end of nineteenth century and led to very important discoveries.

Euclid's fifth postulate plays a fundamental role in elementary geometry because it forms a basis of the theory of parallel lines and the problems connected with it. Recall the result from high school geometry which speaks of the similarity of plane figures and the results of plain trigonometry. Many theorems in high school of geometry consist of comparing sides and angles of two triangles in plane. As we have already mentioned, in the course of composition of such figures one figure is lifted to be placed over another and verify whether an angle (or angles) or a side (sides) of one figure superimposes the respective angle or side of the order. The arguments in support of the process look convincing because they rely heavily in visual evidence rather than logical arguments.

When we talk of sides and angles of a triangle, we are considering the cases of slanted line segments. In case of figures like rectangle, parallelogram and squares we talk of lines which are vertical. All of us know that opposite side of a rectangle are parallel. This definition establishes the existence of parallel lines. But if we go back and try to find the definition of parallel lines, what would we find? We call **two lines parallel if these lines have no point in common**. Two lines have a common point if they intersect at a point. So two lines do not have a common point implies that two parallel lines never intersect. This is the intuitive way of understanding whether two lines are parallel or not.

If two lines are perpendicular to a third line, then the two lines are parallel. Suppose the lines l_1, and l_2 are perpendicular to the line AB. If l_1, and l_2



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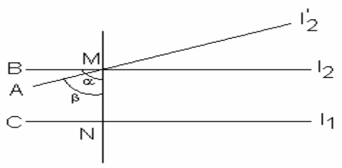
intersect at some point, say C, and then the exterior angle ABC is greater than both the interior angles BAC and BCA. As a consequence, we see that exterior angle ABC is greater than the interior angle BAC, that is, both of these angles are right angles and one right angle (ext ∠ABC) is greater than the other right angle BAC, which is absurd. Another consequence, which is also contrary to the well known result that, "the sum of three angles of a triangle is equal to two right angles" is also violated in this case. We see that in △ABC, ∠ABC and ∠BAC are right angles.
Therefore, $\angle ABC + \angle BAC + \angle ACB = 90^0 + 90^0 + \angle ACB > 180^0$.
This completes the proof of the fact that two lines perpendicular to the third line are parallel to each other.

From the above discussion it is also clear that given a line l_1, and a point M outside this line, only one linel_2, **parallel to l_1**,can be drawn passing through M.



If possible, let l_2 and l'_2 are two lines which are parallel to the given line l_1. Let MN be perpendicular to l_1. Then we have

∠BMN = ∠CNM (interior angles on the same side of MN, because l_2 is parallel to l_1)

∠AMN = ∠CNM (interior angles on the same side of MN, assuming that l'_2 is also parallel to l_2)

Therefore, ∠BMN = ∠AMN.
But it is clear that

$$\angle BMN = \angle BMA + \angle AMN.$$

As a consequence, we see that ∠BMN is equal to ∠AMN as well as greater than ∠AMN, which is absurd.

In the above discussion of parallel lines, considerable use has been made of Euclid's **fifth postulate**.

The fifth postulate is equivalent to the assertion that through any point not lying on a given line there passes only one line parallel to the given line. Many important theorems in Euclidean geometry are based on this assertion. In particular, when two parallel lines intersect a third line, then the corresponding angles are equal and the sum of the interior angles of a triangle is equal to two right angles. Euclid, in his Elements, has not made use of the parallel postulate to prove the first twenty eight propositions.

Euclid might have tried to prove the parallel postulate himself. But the problem to prove it continued from his time till the end of nineteenth century. Many mathematicians tried to prove this parallel postulate but all efforts turned out to be erroneous. All the proofs to prove the fifth postulate introduced some geometrical assertions which slipped into the proofs unnoticed. These geometrical assertions

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looked so simple and obvious that mathematicians used these facts into their proofs before these assertions were established on firm logical ground. Attempts made to prove such assertions without the use of parallel postulate were unsuccessful.

But the attempts made to prove the parallel postulate led to some very valuable results. Logical interdependence of geometrical ideas was discovered and various equivalent statements of parallel postulate were found. We will discuss such discoveries afterwards.

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