KATHMANDU UNIVERSITY JOURNAL OF SCIENCE, ENGINEERING AND TECHNOLOGY
VOL. 9, No. I, July, 2013, pp 175-180

FIXED POINT THEOREMS FOR OCCSAIONALLY WEAKLY COMPATIBLE MAPS IN *G*-METRIC SPACE

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Received 11 May, 2012; Revised 04 February, 2013

ABSTRACT
In this paper, we prove common fixed point theorems for a pair of occasionally weakly compatible maps in Symmetric *G*-metric space. Our results generalize and extend several relevant common fixed point theorems from the literature.

Key words: Symmetric *G*-metric space, occasionally weakly compatible maps, weakly compatible maps.
Subject classification: 2000 AMS: 47H10, 54H25

INTRODUCTION
In 1992, Dhage[1] introduced the concept of D – metric space. Recently, Mustafa and Sims[5] shown that most of the results concerning Dhage's D – metric spaces are invalid. Therefore, they introduced G – metric space. For more details on G – metric spaces, one can refer to the papers [5]-[8].

In 2006, Mustafa and Sims[6] introduced the concept of G-metric spaces as follows:
Definition 1.1.[6] Let X be a nonempty set, and let $G: X \times X \times X \to R^+$ be a function satisfying the following axioms:
(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables) and

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality)
then the function G is called a generalized metric, or, more specifically a G – metric on X and the pair (X, G) is called a G – metric space.
If condition (G6) also satisfied then (X, G) is called Symmetric G-metric space.
(G6) $G(x, y, y) = G(x, x, y)$ for all $x, y \in X$.

Definition 1.2.[6] Let (X, G) be a G–metric space, and let $\{x_n\}$ a sequence of points in X, a point 'x' in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G–convergent to x.

Thus, that if $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ in a G-metric space (X, G) then for each $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq N$.

Proposition 1.1.[6] Let (X, G) be a G – metric space. Then the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x,

(2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$,

(3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$,

(4) $G(x_m, x_n, x) \rightarrow 0$ as $m, n \rightarrow \infty$.

Definition 1.3.[6] Let (X, G) be a G – metric space. A sequence $\{x_n\}$ is called G – Cauchy if, for each $\varepsilon > 0$ there exists a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$; i.e. if $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow \infty$

Proposition 1.2.[6] If (X,G) is a G – metric space then the following are equivalent:

(1) The sequence $\{x_n\}$ is G – Cauchy,

(2) for each $\varepsilon > 0$, there exist a positive integer N such that $G(x_n, x_m, x_m) < \varepsilon$ for all $n, m \geq N$.

Proposition 1.3.[6] Let (X, G) be a G – metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Definition 1.4.[6] A G – metric space (X, G) is said to be G–complete if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 1.4.[6] A G – metric space (X, G) is G – complete if and only if (X, d_G) is a complete metric space.

Proposition 1.5.[6] Let (X, G) be a G – metric space. Then, for any x, y, z, a in X it follows that:

(i) If $G(x, y, z) = 0$, then $x = y = z$,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(y, x, x)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq {}^2/_3\,(G(x, y, a) + G(x, a, z) + G(a, y, z))$,

(vi) $G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

In 1996, Jungck [2] introduced the notion of weakly compatible maps as follows:

Definition 1.5.[2] A pair of self mappings (f, g) of a metric space is said to be weakly compatible if they commute at the coincidence
points i.e. $Tu = Su$ for some u in X, then $TSu = STu$.

Definition 1.6. Let (X, G) be a Symmetric G-metric space. f and g be self maps on X. A point x in X is called a coincidence point of f and g iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of f and g.

Definition 1.7[3]: A pair of self mappings (f, g) of a Symmetric G-metric space (X, G) is said to be weakly compatible if they commute at the coincidence points
i.e., if $fu = gu$ for some u in X, then $fgu = gfu$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 1.8[3]: Two self mappings f and g of a Symmetric G-metric space (X, G) are said to be occasionally weakly compatible (*owc*) iff there is a point x in X which is coincidence point of f and g at which f and g commute.

Lemma 1.1[3]: Let (X, G) be a Symmetric G-metric space. f and g be self maps on X and let f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

MAIN RESULTS

Following to Matkowski[5], let Φ be the set of all functions ϕ such that $\phi:[0,+\infty)\to[0,+\infty)$ be a non-decreasing function with $\lim_{n\to+\infty}\phi^n(t)=0$ for all $t\in[0,+\infty)$. If $\phi\in\Phi$, then ϕ is called Φ- map. If ϕ is Φ- map, then it is an easy matter to show that

(*A*) $\phi(t)<t$ for all $t\in[0,+\infty)$;

(*B*) $\phi(0)=0.$

From now unless otherwise stated, we mean by ϕ the Φ- map. Now, we introduce and prove our result.

Theorem 2.1: Let (X, G) be a Symmetric G-metric space. If f and g are *owc* self maps on X and

$$G(fx,fy,fy) \le \phi\,[max\{G(gx,gy,gy),\ G(gx,fy,fy),\ G(gy,fx,fx),\ G(gy,fy,fy)\}] \qquad (2.1)$$

for all $x, y\in X$. Then f and g have a unique common fixed point.

Proof: Since f and g are *owc*, there exist a point $u\in X$ such that $fu = gu$ and $fgu = gfu$. We claim that fu is the unique common fixed point of f and g. We first assert that fu is a fixed point of f.

For, if $ffu \neq fu$, then from equation (2.1), we get

$$\begin{aligned}G(fu,ffu,ffu) &\le \phi\,[max\{G(gu,gfu,gfu),\ G(gu,ffu,ffu),\ G(gfu,fu,fu),\ G(gfu,ffu,ffu)\}]\\ &= \phi\,[max\{G(\mathrm{fu,ffu,ffu}),\ G(fu,ffu,ffu),\ G(ffu,fu,fu),\ G(ffu,ffu,ffu)\}]\\ &= \phi\,[max\{G(fu,ffu,ffu),\ \mathrm{G}(fu,ffu,ffu),\ \mathrm{G}(fu,fu,ffu),\ 0\}]\\ &= \phi\,[max\{G(fu,ffu,ffu),\ G(fu,ffu,ffu),\ G(fu,ffu,ffu)\}]\\ &= \phi\,[G(fu,ffu,ffu)] < G(fu,ffu,ffu)\end{aligned}$$

a contradiction. So $ffu = fu$ and $ffu = fgu = gfu = fu$. Hence fu is a common fixed point of f and g.

Now we prove uniqueness. Suppose that $u, v\in X$ such that $fu = gu = u$ and $fv = gv = v$ and $u \neq v$. Then from equation (2.1),

$$\begin{aligned}G(u,v,v) = G(fu,fv,fv) &\le \phi\,[max\{G(\mathrm{gu,gv,gv}),\ G(gu,fv,fv),\ G(gv,fu,fu),\ G(gv,fv,fv)\}]\\ &= \phi\,[max\{G(\mathrm{u,v,v}),\ G(\mathrm{u,v,v}),\ \mathrm{G(v,u,u)},\ \mathrm{G(v,v,v)}\}]\\ &= \phi\,[max\{G(u,v,v),\ G(u,v,v),\ G(v,v,u),\ 0\}]\\ &= \phi\,[max\{G(u,v,v),\ G(u,v,v),\ G(\mathrm{u,v,v}),\ 0\}]\\ &= \phi\,[G(u,v,v)] < G(u,v,v)\end{aligned}$$

a contradiction. So $u = v$. Therefore, the common fixed point of f and g is unique.

Theorem 2.2: Let (X, G) be a Symmetric G-metric space. Suppose that f, g, S, T are self maps on X and that the pairs $\{f, S\}$ and $\{g, T\}$ are each *owc*. If

$$G(fx,gy,gy) < max\ \{\ G(Sx,Ty,Ty),\ G(Sx,fx,fx),\ G(Ty,gy,gy),\ G(Sx,gy,gy),\ G(Ty,fx,fx)\ \}, \qquad (2.2)$$

for all $x, y \in X$. Then *f, g, S* and *T* have a unique common fixed point in *X*.

Proof: By hypothesis, there exists points $x, y \in X$ such that $fx = Sx$ and $gy = Ty$. We claim that $fx = gy$. For, otherwise, by (2.2)

$$G(fx,gy,gy) < max \{ G(Sx,Ty,Ty), G(Sx,fx,fx), G(Ty,gy,gy), G(Sx,gy,gy), G(Ty,fx,fx) \}$$
$$= max \{ G(fx,gy,gy), G(fx,fx,fx), G(gy,gy,gy), G(fx,gy,gy), G(gy,fx,fx) \}$$
$$= max \{ G(fx,gy,gy), 0, 0, G(fx,gy,gy), G(gy,gy,fx) \}$$
$$= max \{ G(fx,gy,gy), G(fx,gy,gy), G(fx,gy,gy) \} = G(fx,gy,gy)$$

a contradiction. This implies that $fx = gy$. So $fx = Sx = gy = Ty$. Moreover, if there is another point z such that $fz = Sz$, then, using (2.2) it follows that $fz = Sz = gy = Ty$ or $fx = fz$ and $w = fx = Sx$ is the unique point of coincidence of *f* and *S*. Then by Lemma 1.1, it follows that w is the unique common fixed point of *f* and *S*. By symmetry, there is a unique common fixed point $z \in X$ such that $z = gz = Tz$.

Now, we claim that $w = z$. Suppose that $w \neq z$. Using (2.2),

$$G(w,z,z) = G(fw,gz,gz)$$
$$< max \{ G(Sw,Tz,Tz), G(Sw,fw,fw), G(Tz,gz,gz), G(Sw,gz,gz), G(Tz,fw,fw) \}$$
$$G(w,z,z) < max \{ G(w,z,z), G(w,w,w), G(z,z,z), G(w,z,z), G(z,w,w) \}$$
$$= max \{ G(w,z,z), 0 , 0 , G(w,z,z), G(z,z,w) \}$$
$$= max \{ G(w,z,z), G(w,z,z), G(w,z,z) \} = G(w,z,z)$$

This is a contradiction. Therefore $w = z$ and w is a unique point of coincidence of *f, g, S* and *T*. By Lemma 1.1, *w* is the unique common fixed point of *f, g, S* and *T*.

Corollary 2.1: Let (*X*, *G*) be a Symmetric *G*-metric space. Suppose that *f, g, S* and *T* are self maps on *X* and that the pairs {*f* , *S*} and {*g* , *T*} are each *owc*. If

$G(fx,gy,gy) \leq h\, m(x,y,y)$ where

$$m(x,y,y) = max\{G(Sx,Ty,Ty),G(Sx,fx,fx),G(Ty,gy,gy),[G(Sx,gy,gy),G(Ty,fx,fx)]/2\}, \quad (2.3)$$

for all $x, y \in X$ and $0 \leq h < 1$, then *f, g, S* and *T* have a unique common fixed point in *X*.

Proof: Since (2.3) is a special case of (2.2), the result follows immediately from Theorem 2.2.

Theorem 2.3. Let *A, B, S* and *T* be self maps of Symmetric *G*-metric space (*X*, *G*), satisfying the following conditions:

(2.4) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(2.5) pairs (*A*, *S*) or (*B, T*) satisfies property *E.A.*,

(2.6) for all $x, y \in X$,

$$G(Ax, By, By) < \phi [\, max \{G(Sx, Ty, Ty), G(Sx, By, By), G(Ty, By, By)\}]$$

where $\phi \in \Phi$. If one of *A(X), B(X), S(X)* or *T(X)* is complete subsets of *X* then pairs (*A*, *S*) and (*B, T*) have coincidence point.

Further, if (*A*, *S*) and (*B, T*) are weakly compatible then *A, B, S* and *T* have unique common fixed point in *X*.

Proof: Suppose the pair (*B, T*) satisfies the property (*E.A.*). Then there exists a sequence $\{x_n\}$ in *X* such that

$lim_{n\to\infty}Bx_n = lim_{n\to\infty}Tx_n = p$ for some $p \in X$.

Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in *X* such that

$Bx_n = Sy_n = p$. Hence $lim_{n\to\infty}Sy_n = p$.

We shall show that $lim_{n\to\infty}Ay_n = p$.

From (2.6), we have

$$G(Ay_n, Bx_n, Bx_n) < \phi [\, max \{G(Sy_n, Tx_n, Tx_n), G(Sy_n, Bx_n, Bx_n), G(Tx_n, Bx_n, Bx_n)\}]$$

Taking limit as $n \to \infty$, we get

$lim_{n\to\infty} G(Ay_n, p, p) < \phi[max\{G(p, p, p), G(p, p, p), G(p, p, p)\}]$

$= \phi[max\{0, 0, 0\}] = \phi(0) = 0.$

This implies, $lim_{n\to\infty} Ay_n = p$.

Thus we have, $lim_{n\to\infty} Ay_n = lim_{n\to\infty} Sy_n = lim_{n\to\infty} Bx_n = lim_{n\to\infty} Tx_n = p$.

Suppose that $S(X)$ is a complete subspace of X. Then $p = Su$ for some $u \in X$. Subsequently, we have

$lim_{n\to\infty} Ay_n = lim_{n\to\infty} Sx_n = lim_{n\to\infty} Bx_n = lim_{n\to\infty} Tx_n = p = Su$

Now, we shall show that $Au = Su$.

From (2.6), we have

$G(Au, Bx_n, Bx_n) < \phi[max\{G(Su, Tx_n, Tx_n), G(Su, Bx_n, Bx_n), G(Tx_n, Bx_n, Bx_n)\}]$

Taking limit as $n \to \infty$ we get

$$G(Au, Su, Su) < \phi[max\{G(p, p, p), G(p, p, p), G(p, p, p)\}]$$
$$= \phi[max\{0, 0, 0\}] = \phi(0) = 0.$$

Thus, we have $Au = Su$. Therefore (A, S) have coincidence point.

The weak compatibility of A and S implies that $ASu = SAu$ and thus $AAu = ASu = SAu = SSu$.

As $A(X) \subset T(X)$, there exists $v \in X$ such that $Au = Tv$. We claim that $Tv = Bv$.

Suppose not, from (2.6) , we have

$$G(Au, Bv, Bv) < \phi[max\{G(Su, Tv, Tv), G(Su, Bv, Bv), G(Tv, Bv, Bv)\}]$$
$$= \phi[max\{0, G(Au, Bv, Bv), G(Au, Bv, Bv)\}]$$
$$= \phi[G(Au, Bv, Bv)] < G(Au, Bv, Bv),$$

this implies, $Au = Bv$.

Hence, $Tv = Bv$. Therefore (B, T) have coincidence point

Thus we have $Au = Su = Tv = Bv$.

The weak compatibility of B and T implies that $BTv = TBv = TTv = BBv$.

Finally, we show that Au is the common fixed point of A, B, S and T.

From (2.6), suppose $Au \neq AAu$, we have

$G(Au, AAu, AAu) = G(Au, Au, AAu)$ { by definition of symmetric space}

$= G(AAu, Bv, Bv) < \phi[max\{G(SAu, Tv, Tv), G(SAu, Bv, Bv), G(Tv, Bv, Bv)\}]$

$$= \phi[max\{G(AAu, Bv, Bv), G(AAu, Bv, Bv), G(Bv, Bv, Bv)\}]$$
$$= \phi[max\{G(AAu, Bv, Bv), G(AAu, Bv, Bv), 0\}]$$
$$= \phi[G(AAu, Bv, Bv)] < G(AAu, Bv, Bv),$$

This gives, $AAu = Bv = Au$ and thus $AAu = Au$.

Therefore, $Au = AAu = SAu$ is the common fixed point of A and S.

Similarly, we prove that Bv is the common fixed point of B and T. Since $Au = Bv$, Au is common fixed point of A, B, S and T. The proof is similar when $T(X)$ is assumed to be a complete subspace of X. The cases in which $A(X)$ or $B(X)$ is a complete subspace of X are similar to the cases in which $T(X)$ or $S(X)$, respectively is complete subspace of X as $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

Finally now we show that the common fixed point is unique. If possible, let x_0 and y_0 be two common fixed points of A, B, S and T. Suppose $x_0 \neq y_0$, then by condition (2.6), we have

$G(x_0, y_0, y_0) = G(Ax_0, By_0, By_0)$

$$< \phi[max\{G(Sx_0, Ty_0, Ty_0), G(Sx_0, By_0, By_0), G(Ty_0, By_0, By_0)\}]$$

$= \phi\,[max\,\{G(x_0, y_0, y_0), G(x_0, y_0, y_0), G(y_0, y_0, y_0)\}]$
$= \phi\,[G(x_0, y_0, y_0)] < G(x_0, y_0, y_0)$,

this implies $x_0 = y_0$.
Therefore, the mappings *A, B, S* and *T* have a unique common fixed point.

Corollary 2.2. Let *A*, *B* and *S* be self maps of Symmetric *G*-metric space (*X*, *G*), satisfying the following conditions:

(2.7) $A(X) \subset S(X), B(X) \subset S(X)$,
(2.8) pairs (*A*, *S*) or (*B*, *S*) satisfies property *E.A.*,
(2.9) for all $x, y \in X$,

$$G(Ax, By, By) < \phi\,[\max\{G(Sx, Sy, Sy), G(Sx, By, By), G(Sy, By, By)\}]$$

where $\phi \in \Phi$. If one of *A*(*X*), *B*(*X*) or *S*(*X*) is complete subsets of *X* then pairs (*A*, *S*) and (*B*, *S*) have coincidence point.
Further, if (*A*, *S*) and (*B*, *S*) are weakly compatible then *A*, *B* and *S* have unique common fixed point in *X*.

Proof: Take *T* = *S* in Theorem 2.3.

ACKNOWLEDGEMENT

The author is grateful to an anonymous referee for his fruitful suggestions.

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