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COMMON FIXED POINT THEOREM ON CONE BANACH SPACE

[1]P. G. Varghese *, [2]K. S. Dersanambika

[1] Department of Mathematics, Marthoma College, Thiruvalla – 689103, Kerala, India.
[2] Department of Mathematics, Fatima Mata National College, Kollam-691001, Kerala, India

* Corresponding address: pg.varghese@ymail.com
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ABSTRACT
The aim of this manuscript is to discus some fixed point results for self mappings satisfying certain contractive conditions on Cone Banach space.

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INTRODUCTION
In 2007, Huang and Zhang [4] introduced the notion of cone metric space, replacing the set of real numbers by ordered Banach space and proved some fixed point theorems for functions satisfying contractive conditions in these spaces. The results in [4] were generalized by Sh. Rezapour and R. Hamlbarani [7] by omitting the normality condition, which is a mile stone in developing fixed point theory in cone metric space. After that several articles on fixed point theorems in cone metric space were obtained by different mathematicians such as M.Abbas, G. Junck [1], D.Ilic [5] etc. Very recently some results on fixed point theorems have been extended to Cone Banach space. E. Karapinar [8] proved some fixed point theorems for self mappings satisfying some contractive condition on a Cone Banach space. Thabet Abdeljawad, E. Karapinar and Kenan Tas [9] have given some generalizations to this theorems. Neeraj Malviya and Sarala Chouhan [6] extended some fixed point theorems to Cone Banach space.

DEFINITION 1.1. [4] Let $(E, \|.\|)$ be a real Banach space. A subset $P \subseteq E$ is said to be a cone if and only if

(1) P is closed, nonempty and $P \neq \{0\}$

(2) a, b $\in R$, $a, b \geq 0, x, y \in P$ implies $ax + by \in P$

(3) $P \cap (-P) = \{0\}$

For a given cone P subset of E, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in$ int P where int P denotes interior of P and is assumed to be nonempty.

DEFINITION 1.2. [4] Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfies

(1) $0 \leq d(x, y)$ for every $x, y \in X$, $d(x, y) = 0$ if and only if x = y.

(2) $d(x, y) = d(y, x)$ for every $x, y \in X$

(3) $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \in X$

Then d is a cone metric on X and (X, d) is a cone metric space.

DEFINITION 1.3. [8] Let X be a vector space over R. Suppose the mapping $\|.\| : X \to E$ satisfies

(1) $\|x\| > 0$ for all $x \in X$

(2) $\|x\| = 0$ if and only if $x = 0$

(3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

(4) $\|kx\| = |k| \|x\|$ for all $k \in R$

Then $\|.\|$ is called a norm on X, and $(X, \|.\|)$ is called a cone normed space. Clearly each cone normed space if a cone metric space with metric defined by d (x, y) = $\|x - y\|$.

DEFINITION 1.4. [8] Let $(X, \|.\|)$ be a cone normed space, x $\in$ X and $\{x_n\}$ a sequence in X. Then

(1) $\{x_n\}$ converges to x if for every c $\in$ E with o << c there is a natural number N such that $\|x_n - x\| \leq c$ for all $n \geq N$. We shall denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(2) $\{x_n\}$ is a Cauchy sequence, if for every $c \in E$ with $o << c$ there is a natural number N such that $\|x_n - x_m\| \leq c$ for all $n, m \geq N$.

(3) $(X, \|.\|)$ is a complete cone normed space if every Cauchy sequence is convergent.

A complete cone normed space is called a Cone Banach space.

DEFINITION 1.5. Let f and g be self mappings on a cone normed space $(X, \|.\|)$, they are said to be compatible if $\lim_{n\to\infty} \|fgx_n - gfx_n\| = 0$ for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} fx_n - \lim_{n\to\infty} gx_n = y$ for some point y in X.

PROPOSITION 1.6. [8] Let $(X, \|.\|)$ be a cone normed space. P be a normal cone with constant K. Let $\{x_n\}$ be a sequence in X. Then

(1) $\{x_n\}$ converges to x if and only if $\|x_n - x\| \to 0$ as $n \to \infty$.

(2) $\{x_n\}$ is a Cauchy sequence if and only is $\|x_n - x_m\| \to 0$ as $n, m \to \infty$

(3) if the $\{x_n\}$ converges to x and $\{y_n\}$ converges to y then $\|x_n - y_n\| \to \|x - y\|$

PROPOSITION 1.7. Let f and g be compatible mappings on a cone normed space $(X, \|.\|)$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = y$ for some point y in X and for every sequence $\{x_n\}$ in X. Then $\lim_{n\to\infty} gfx_n = fy$ if f is continuous.

RESULTS

THEOREM 2.1. Let (X, $\|.\|$) be a Cone Banach space, P is a cone in E, with $\|x\|$ = d(x, 0), f and g be self mappings on X satisfying

$$\|fx - fy\| \leq a\|gx - gy\| + b\|fx - gx\| + c\|fy - gy\| + d\|fx - gy\| + e\|fy - gx\| \quad (1)$$

where a + b + c + 2d + e $\in$ [0, 1), x, y, $\in$ X. If the range of g contains the range of f, g(X) is complete, f; g are continuous and compatible. Then f and g have a unique common fixed point.

PROOF : Let $x_0 \in X$, since $f(x_0) \in f(X) \subseteq g(X)$ there is $x_1 \in X$ such that $f(x_0) = g(x_1)$. Let $y_1 = f(x_0) = g(x_1)$. Now $x_1 \in X$, since $f(x_1) \in f(X) \subseteq g(X)$ there is an x_2 such that $f(x_1) = g(x_2)$. Let $y_2 = f(x_1) = g(x_2)$ continuing like this we get a sequence $\{y_n\}$ in X such that $y_n = f(x_{n-1}) = g(x_n), n = 0,1,2........$

We shall prove

$$\| y_n - y_{n-1} \| \le \lambda \| y_{n-1} - y_{n-2} \|$$

Where $\lambda < 1$. We have

$$\begin{aligned} \| y_n - y_{n-1} \| &= \| g(x_{n+1}) - g(x_n) \| \\ &= \| f(x_n) - f(x_{n-1}) \| \\ &\le a \| g(x_n) - g(x_{n-1}) \| + b \| f(x_n) - g(x_n) \| \\ &\quad + c \| f(x_{n-1}) g(x_{n-1}) \| + d \| f(x_n) - g(x_{n-1}) \| + e \| fx_{n-1} - g(x_n) \| \\ &= a \| y_{n-1} - y_{n-2} \| + b \| y_n - y_{n-1} \| + c \| y_{n-1} - y_{n-2} \| \\ &\quad + d \| y_n - y_{n-2} \| + e \| y_{n-1} - y_{n-1} \| \end{aligned}$$

$$\begin{aligned} &= a \| y_{n-1} - y_{n-2} \| + b \| y_n - y_{n-1} \| + c \| y_{n-1} - y_{n-2} \| \\ &\quad + d \| y_n - y_{n-1} \| + e \| y_{n-1} - y_{n-2} \| \end{aligned}$$

$(1- b - d) \| y_n - y_{n-1} \| \le (a + c + d) \| y_{n-1} - y_{n-2} \|$

$$\| y_n - y_{n-1} \| \le \frac{a+c+d}{1-b-d} \| y_{n-1} - y_{n-2} \|$$

Let $\lambda = \frac{a+c+d}{1-b-d}$, then $\lambda < 1$

Therefore

$$\| y_n - y_{n-1} \| \le \| y_{n-1} - y_{n-2} \|, \text{ for every n}$$

Hence

$$\| y_n - y_{n-1} \| \le \lambda^{n-1} \| y_1 - y_0 \|$$

Let n > m

$$\begin{aligned} \| y_n - y_m \| &\le \| y_n - y_{n-1} \| + \| y_{n-1} - y_{n-2} \| + ... + \| y_{n-1} - y_m \| \\ &\le \left(\lambda^{n-1} + \lambda^{n-2} + ... + \lambda^m \right) \| y_1 - y_0 \| \end{aligned}$$

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$$\leq \frac{\lambda^m}{1-\lambda} \| y_1 - y_0 \| \qquad (2)$$

Let c $\geq$ 0, then there is a $\delta > 0$ such that $c + N_\delta(0) \subseteq P$ where $N_\delta(0) = \{y : \| y \| < \delta\}$.

Since $\lambda < 1$, for $\delta \geq 0$ there is a positive integer N such that $\frac{\lambda^m}{1-\lambda} \| y_1 - y_0 \| < \delta$ for $m \geq$ N.

Hence $\frac{\lambda^m}{1-\lambda} \| y_1 - y_0 \| \in N_\delta(0)$. So $-\frac{\lambda^m}{1-\lambda} \| y_1 - y_0 \| \in N_\delta(0)$.

Therefore $c - \frac{\lambda^m}{1-\lambda} \| y_1 - y_0 \| \in c + N_\delta(0) \subseteq P$. This implies $\frac{\lambda^m}{1-\lambda} \| y_1 - y_0 \| \leq c$. Therefore (2) becomes $\| y_n - y_m \| \leq c$ for n, m $\geq$ N. Hence $\{y_n\}$ is a Cauchy sequence in g(X). But g(X) is complete therefore there is an $z \in g(X)$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_{n+1}) = z.$

Since f and g are continuous $\lim_{n\to\infty} f^2(x_n) = f(z),\ \lim_{n\to\infty} g^2(x_n) = g(z),\ \lim_{n\to\infty} gf(x_n) = g(z),$ $\lim_{n\to\infty} fg(x_n) = f(z)$.

Now we shall prove f(z) = g(z). We have

$$\| f^2(x_n) - gf(x_n) \| = \| f^2(x_n) - fg(x_n) \|$$

$$\leq a \| gf(x_n) - g^2(x_n) \| + b \| f^2(x_n) - gf(x_n) \| +$$

$$c \| fg(x_n) - g^2(x_n) \| + d \| f^2(x_n) - g^2(x_n) \| + e \| fg(x_n) - gf(x_n) \|$$

Letting $n \to \infty$. We get

$$\| f(z) - g(z) \| \leq b \| f(z) - g(z) \| + c \| f(z) - g(z) \| +$$

$$d \| f(z) - g(z) \| + e \| f(z) - g(z) \|$$

$$(1 - b - c - d - e) \| f(z) - g(z) \| \leq 0$$

But (1- b – c – d - e) > 0. Therefore $\| f(z) - g(z) \| = 0$ which implies f(z) = g(z). Hence z is a point of coincidence of f and g. To prove uniqueness if possible z_1 is another point of coincidence of f and g. Then $y_1 = f(z_1) = g(z_1)$.

Now

$$\| f(z) - f(z_1) \| \leq a \| g(z) - g(z_1) \| + b \| f(z) - g(z) \| +$$

$$c \| f(z_1) - g(z_1) \| + d \| f(z) - g(z_1) \| + e \| f(z_1) - g(z) \|$$

$$= a \| f(z) - f(z_1) \| + d \| f(z) - f(z_1) \| + e \| f(z_1) - f(z) \|$$

$(1 - a - d - e) \| f(z) - f(z_1) \| \leq 0$

But (1 - a - d - e) > 0. Hence $f(z) = f(z_1)$. So $f(z) = g(z) = f(z_1) = g(z_1)$, the uniqueness of z. Since f and g are compatible by definition 2.5, f and g have a unique common fixed point.

THEOREM 2.2 Let f, g, h, l be mappings on Cone Banach space $(X, \|.\|)$ into itself, with $\|x\| = d(x, 0)$ satisfying the conditions.

(1) $\| hx - ly\| \leq a \| fx - hx\| + b \|fx-ly\| + c \| gy - ly\|$ for all $x, y \in X$, $a, b, c \geq 0$, $a+2b+c < 1$.

(2) f and g are onto mapping

(3) f is continuous

(4) f and h; g and l commute

Then f, g, h and l have a unique common fixed point.

PROOF: Let $x_0 \in X$ be arbitrary, then $l(x_0) \in X$, since f is onto there is an $x_1 \in X$ such that $f(x_1) = l(x_0)$. Let $y_0 = f(x_1)= l(x_0)$. Again $x_1 \in X$ since g is onto there is a $x_2 \in X$ such that $g(x_2) = h(x_1)$. Let $y_2 = g(x_2) = h(x_1)$. Continuing like this we get a sequence $\{y_n\}$ such that

$y_{2n} = f(x_{2n+1}) = l(x_{2n})$ and $y_{2n+1} = g(x_{2n+2}) = h(x_{2n+1})$. We have

$$\| y_{2n-1} - y_{2n} \| = \| h(x_{2n-1}) - l(x_{2n}) \|$$

$$\leq a \| f(x_{2n-1}) - h(x_{2n-1}) \| + b \| f(x_{2n-1}) - l(x_{2n}) \| + c \| g(x_{2n}) - h(x_{2n}) \|$$

$$\| y_{2n} - y_{2n-1} \| \leq a \| y_{2n-2} - y_{2n-1} \| + b \| y_{2n-2} - y_{2n} \| + c \| y_{2n-1} - y_{2n} \|$$

$$\leq a \| y_{2n-2} - y_{2n-1} \| + b \| y_{2n-2} - y_{2n-1} \|$$

$$+ b \| y_{2n-1} - y_{2n} \| + c \| y_{2n-1} - y_{2n} \|$$

$$(1 - b - c) \|y_{2n-1} - y_{2n}\| \leq (a+b) \| y_{2n-2} - y_{2n-1} \|$$

$$\| y_{2n-1} - y_{2n} \| \leq \frac{a+b}{1-b-c} \| y_{2n-2} - y_{2n-1} \|$$

Let $\lambda_1 = \dfrac{a+b}{1-b-c}$, $\lambda_1 < 1$. Therefore

$$\| y_{2n-1} - y_{2n} \| \leq \lambda_1 \| y_{2n-2} - y_{2n-1} \| \qquad (3)$$

$$\| y_{2n} - y_{2n+1} \| = \| hx_{2n+1} - lx_{2n} \|$$

$$\leq a \| fx_{2n+1} - hx_{2n+1} \| + b \| fx_{2n+1} - lx_{2n} \| + c \| gx_{2n} - lx_{2n} \|$$

$$\| y_{2n} - y_{2n+1} \| \leq a \| y_{2n} - y_{2n+1} \| + b \| y_{2n} - y_{2n} \| + c \| y_{2n-1} - y_{2n} \|$$

$$(1-a) \| y_{2n} - y_{2n+1} \| \leq c \| y_{2n-1} - y_{2n} \|$$

Let $\lambda_2 = \dfrac{c}{1-a}$, then $\lambda_2 < 1$. Therefore

$$\| y_{2n} - y_{2n+1} \| \leq \lambda_2 \| y_{2n-1} - y_{2n} \| \qquad (4)$$

Let $\lambda = \max(\lambda_1, \lambda_2)$ then $\lambda < 1$, (6) and (7) becomes

$$\| y_{2n-1} - y_{2n} \| \leq \lambda \| y_{2n-2} - y_{2n-1} \| \qquad (5)$$

And

$$\| y_{2n} - y_{2n+1} \| \leq \lambda \| y_{2n-1} - y_{2n} \| \qquad (6)$$

From (5) and (6) we get

$$\|y_n - y_{n+1}\| \le \lambda^n \|y_1 - y_0\| \qquad (7)$$

For every n. Let $n > m$

$$\begin{aligned}\|y_n - y_m\| &\le \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \ldots\ldots + \|y_{n-1} - y_n\| \\ &\le \left(\lambda^m + \lambda^{m+1} + \ldots\ldots + \lambda^{n-1}\right)\|y_1 - y_0\| \\ &< \frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \qquad (8)\end{aligned}$$

Let c > 0, then there is a $\delta > 0$ such that $c + N_\delta(0) \subseteq P$ where $N_\delta(0) = \{y \in E : \|y\| \le \delta\}$ .Since $\lambda < 1$ there exist a positive integer N such that $\frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \le \delta$ for every $m \ge N$. Hence $\frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \in N_\delta(0)$, which implies $-\frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \in N_\delta(0)$.

Therefore $c - \frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \in c + N_\delta(0) \subseteq P$. This implies $\frac{\lambda^m}{1-\lambda}\|y_1 - y_0\| \le c$ for n, m $\ge$ N. Hence by (8) $\|y_n - y_m\| \le c$ for every $n, m \ge N$ so by definition 2.4 $\{y_n\}$ is a Cauchy sequence in X. Since X is complete there is an z $\in$ X such that $y_n \to z$. Therefore $\{f(x_{2n+1})\}, \{g(x_{2n})\}, \{h(x_{2n+1})\}, \{l(x_{2n})\}$ converges to z. Continuity of f implies $f^2(x_{2n+1}) \to f(z)$. Since h and f commute $hf(x_{2n+1}) = fh(x_{2n+1}) \to f(z)$. We have

$$\begin{aligned}\|hf(x_{2n+1}) - l(x_{2n})\| \le\ & a\|f^2(x_{2n+1}) - hf(x_{2n+1})\| + \\ & b\|f^2(x_{2n+1}) - l(x_{2n})\| + c\|g(x_{2n}) - l(x_{2n})\|\end{aligned}$$

Letting $n \to \infty$ we get

$$\begin{aligned}\|f(z) - z\| &\le a\|f(z) - f(z)\| + b\|f(z) - z\| + c\|z - z\| \\ &(1-b)\|f(z) - z\| \le 0\end{aligned}$$

But 1- b > 0 so || f(z) – z || = 0 which implies f(z) = z. Again

$$\|h(z) - l(x_{2n})\| \le a\|f(z) - h(z)\| + \|bf(z) - l(x_{2n})\| + c\|g(x_{2n}) - l(x_{2n})\|$$

Letting $n \to \infty$ we get

$$\begin{aligned}&\|h(z) - z\| \le a\|z - h(z)\| + b\|z - z\| + c\|z - z\| \\ &(1-a)\|h(z) - z\| \le 0\end{aligned}$$

But 1-a > 0, hence h(z) = z. Since g is onto there is u $\in$ X, such that z = g(u). We have

$$\begin{aligned}\|hf(x_{2n+1}) - l(u)\| \le\ & a\|f^2(x_{2n+1}) - hf(x_{2n+1})\| + b\|f^2(x_{2n+1}) - l(u)\| + \\ & c\|g(u) - l(u)\|\end{aligned}$$

Letting $n \to \infty$ we get

$$\| z - l(u) \| \leq a \| z - z \| + b \| z - l(u) \| + c \| z - l(u) \|$$

$$(1 - b - c) \| z - l(u) \| \leq 0$$

Since 1-b-c > 0, this implies z = l(u). Therefore g(u) = l(u) = z. Since g and l commute l(z) = lg (u) = gl (u) = g(z). We have

$$\| h(x_{2n+1} - l(z) \| \leq a \| f(x_{2n+1}) - h(x_{2n+1}) \| + b \| f(x_{2n+1}) - l(z) \| + c \| g(u) - l(u) \|$$

Letting $n \to \infty$ we get

$$\| z - l(z) \| \leq a \| z - z \| + b \| z - l(z) \| + c \| z - l(z) \|$$

$$(1 - b - c) \| z - l(z) \| \leq 0$$

But 1- b - c > 0, hence z = l(z). This implies g(z) = l(z) = z. Therefore f (z) = g(z) = h(z) = l(z) = z. Hence z is a fixed point of f, g, h and l.

To prove the uniqueness let z' be another fixed point of f, g, h and l.

$$\| z - z' \| = \| h(z) - l(z) \|$$

$$\leq a \| f(z) - h(z) \| + b \| f(z) - l(z') \| + c \| g(z') - l(z') \|$$

$$= a \| z - z \| + b \| z - z' \| + c \| z - z' \|$$

(1-b - c) $\| z - z' \| \leq 0$

But 1- b - c > 0 thus $z = z'$. Therefore f, g, h and l has a unique common fixed point.

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