A FIXED POINT THEOREM FOR SEMI COMPATIBLE MAPS IN FUZZY METRIC SPACE

Kanhaiya Jha

School of Science, Kathmandu University, P.O. Box No. 6250, Kathmandu, NEPAL

Corresponding author: jhakn(at)ku.edu.np

ABSTRACT

The aim of the present paper is to establish a common fixed point theorem for semi-compatible pair of self maps in a fuzzy metric space which generalizes and improves various well-known comparable results.

1. INTRODUCTION

The study of common fixed points of mappings in a fuzzy metric space satisfying certain contractive conditions has been at the center of vigorous research activity. The concept of fuzzy sets was initiated by Zadeh [27] in 1965. With the concept of fuzzy sets, the fuzzy metric space was introduced by Kramosil and Michalek [12]. Grabiec [7] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by Subrahmanyam [25] for a pair of commuting mappings. Also, George and Veeramani [6] modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. In 1999, Vasuki [26] introduced the concept of R-weak commutativity of mappings in fuzzy metric space and Pant [16] introduced the notion of reciprocal continuity of mappings in metric spaces. Also, Jungck and Rhoades [10] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. Balasubramaniam *et.al.* [1] proved a fixed point theorem, which generalizes a result of Pant for fuzzy mappings in fuzzy metric space. Pant and Jha [17] proved a fixed point theorem that gives an analogue of the results by Balasubramaniam *et.al.*[1] by obtaining a connection between the continuity and reciprocal continuity for four mappings in fuzzy metric space.

Recently, Kutukcu *et.al.* [13] has established a common fixed point theorem in a fuzzy metric space by studying the relationship between the continuity and reciprocal continuity which is a generalization of the results of Mishra [14] and also gives an answer to the open problem of Rhoades [19] in fuzzy metric space. Jha *et.al.*[9] has proved a common fixed point theorem for four self mappings in fuzzy metric space under the weak contractive conditions. Also, B. Singh and S. Jain [23] introduced the notion of semi-compatible maps in fuzzy metric space and compared this notion with the notion of compatible map, compatible map of type (α), compatible map of type (β) and obtained some fixed point theorems in complete fuzzy metric space in the sense of Grabiec [7]. As a generalization of fixed point results of Singh and Jain [23], Mishra

et. al.[15] proved a fixed point theorems in complete fuzzy metric space by replacing continuity condition with reciprocally continuity maps.

The purpose of this paper is to obtain a common fixed point theorem for semicompatible pair of self mappings in fuzzy metric space. Our result generalizes and improves various other similar results of fixed points. We also give an example to illustrate our main theorem.

We have used the following notions:

DEFINITION 1.1([27]) Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1].

DEFINITION 1.2([6]) A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if, $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in [0, 1].

For an example: a * b = ab, a * b = min {a, b}.

DEFINITION 1.3([6]) The triplet (X, M, $*$) is called a fuzzy metric space (shortly, a FM-space) if, X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, 1)$ satisfying the following conditions: for all x, y, z in X, and s, t > 0,
(i) $M(x, y, 0) = 0$, $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ for all t > 0 if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous.
In this case, M is called a fuzzy metric on X and the function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.
Also, we consider the following condition in the fuzzy metric space (X, M, $*$):
(vi) $\lim_{t \to \infty} M(x, y, t) = 1$, for all $x, y \in X$.
It is important to note that every metric space (X, d) induces a fuzzy metric space (X, M, $*$) where $a * b = \min \{a, b\}$ and for all a, b $\in$ X, we have $M(x, y, t) = \dfrac{t}{t + d(x, y)}$, for all t > 0, and $M(x, y, 0) = 0$, so-called the fuzzy metric space induced by the metric d.

DEFINITION 1.4([6]) A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is called a Cauchy sequence if, $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ for every t > 0 and for each p > 0.
A fuzzy metric space (X, M, $*$) is complete if, every Cauchy sequence in X converges in X.

DEFINITION 1.5([6]) A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is said to be convergent to x in X if, $\lim_{n \to \infty} M(x_n, x, t) = 1$, for each t > 0.
It is noted that since $*$ is continuous, it follows from the condition (iv) of Definition (1.3.) that the limit of a sequence in a fuzzy metric space is unique.

DEFINITION 1.6([1]) Two self mappings A and S of a fuzzy metric space (X, M, ∗) are said to be compatible if, $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$, for some p in X.

DEFINITION 1.7([10]) Two self mappings A and S of a fuzzy metric space (X, M, ∗) are said to be weakly compatible if, they commute at coincidence points. That is, $Ax = Sx$ implies that $ASx = SAx$ for all x in X. It is important to note that compatible mappings in a metric space are weakly compatible but weakly compatible mappings need not be compatible [24].

DEFINITION 1.8([23]) Two self mappings A and S of a fuzzy metric space (X, M, ∗) are said to be semi-compatible if, $\lim_{n\to\infty} M(ASx_n, Sx, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$, for some p in X.

DEFINITION 1.9([16]) Two self mappings A and S of a fuzzy metric space (X, M, ∗) are said to be reciprocally continuous if, $\lim_{n\to\infty} M(ASx_n, Ax, t) = 1$ and $\lim_{n\to\infty} M(SAx_n, Sx, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$, for some p in X.
It is noted that if A and S are both continuous, they are obviously reciprocally continuous but the converse need not be true. For this, we have the following example:

EXAMPLE 1.10([16]) Consider two mappings A and S defined on X = [2, 20] with the usual Euclidean metric d, defined by the rule A2 = 2, $Ax = 3$ if $x > 2$ and S2 = 2, $Sx = 6$ if $x > 2$. Then, A and S are reciprocally continuous but are not continuous.

LEMMA 1.11([20]) Let (X, M, ∗) be a fuzzy metric space. If there exists k ∈ (0, 1) such that $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

LEMMA 1.12([3]) Let A and S be two self maps on a fuzzy metric space (X, M, ∗). If the pair (A, S) is reciprocally continuous, then (A, S) is semi-compatible if, and only if (A, S) is compatible.

If A, B, S and T are self mappings of fuzzy metric space (X, M, ∗) in the sequel, we shall denote
$N(x, y, t) = \min\{M(Ax, Sx, t), M(By, Ty, t), M(Sx, Ty, t), M(Ax, Ty, \alpha t),$
$$M(Sx, By, (2 - \alpha)t)\}, \text{ for all } x, y \in X, \alpha \in (0, 2) \text{ and } t > 0.$$

2. MAIN RESULTS

THEOREM 2.1. Let (X, M, ∗) be a complete fuzzy metric space with additional condition (vi) and with $a \ast a \geq a$ for all $a \in [0, 1]$. Let A, B, S and T be mappings from X into itself such that
   (i) $AX \subseteq TX, BX \subseteq SX$, and
   (ii) $M(Ax, By, t) \geq r(N(x, y, t))$,
where r : [0, 1] → [0, 1] is a continuous function such that r(t) > t for some 0 < t < 1 and for all x, $y \in X$, $\alpha \in (0, 2)$ and t > 0. If (A, S) or (B, T) is semi-compatible pair of reciprocally continuous maps with respectively (B, T) or (A, S) as weakly compatible maps, then A, B, S and T have a unique common fixed point in X.

PROOF:

Let $x_0 \in X$ be an arbitrary point. Then, since $AX \subseteq TX$, $BX \subseteq SX$, there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for n = 0, 1, 2, ...
Now, we put $\alpha = 1 - q$ with $q \in (0, 1)$ in (ii), then we have

$M(y_{2n}, y_{2n+1}, t) = M(Ax_{2n}, Bx_{2n+1}, t) \geq r(\min\{M(Ax_{2n}, Sx_{2n}, t), M(Bx_{2n+1}, Tx_{2n+1}, t),$
$$M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1}, (1 - q)t), M(Sx_{2n}, Bx_{2n+1}, (1 + q)t)\}).$$

That is,

$M(y_{2n}, y_{2n+1}, t) \geq r(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t),$
$$M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n+1}, (1 + q)t)\})$$
$$\geq r(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n+1}, qt)\})$$
$$\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n}, y_{2n+1}, qt).$$

Since t-norm $*$ is continuous, letting $q \rightarrow 1$, we have

$M(y_{2n}, y_{2n+1}, t) \geq r(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\}$
$$\geq r(\min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n+1}, t)\}).$$

It follows that, $M(y_{2n}, y_{2n+1}, t) > M(y_{2n-1}, y_{2n}, t)$, since $r(t) > t$ for each $0 < t < 1$.
Similarly, $M(y_{2n+1}, y_{2n+2}, t) > M(y_{2n}, y_{2n+1}, t)$. Therefore, in general, we have
$M(y_n, y_{n+1}, t) \geq r(M(y_{n-1}, y_n, t)) > M(y_{n-1}, y_n, t)$.

Therefore, $\{M(y_n, y_{n+1}, t)\}$ is an increasing sequence of positive real numbers in [0, 1] and tends to a limit, say $\lambda \leq 1$. We claim that $\lambda = 1$. If $\lambda < 1$, then $M(y_n, y_{n+1}, t) \geq r(M(y_{n-1}, y_n, t))$.
So, on letting $n \rightarrow \infty$, we get $\lim_{n\rightarrow\infty} M(y_n, y_{n+1}, t) \geq r(\lim_{n\rightarrow\infty} M(y_n, y_{n+1}, t))$,
that is, $\lambda \geq r(\lambda) > \lambda$, a contradiction. Thus, we have $\lambda = 1$.
Now, for any positive integer p, we have
$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t) * M(y_{n+1}, y_{n+2}, t/p) * ... * M(y_{n+p-1}, y_{n+p}, t/p)$.
Letting $n \rightarrow \infty$, we get $\lim_{n\rightarrow\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * ... * 1 = 1$.
Thus, we have $\lim_{n\rightarrow\infty} M(y_n, y_{n+p}, t) = 1$. Hence, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete metric space, so the sequence $\{y_n\}$ converges to a point u (say) in X and consequently, the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converges to u.

We first consider the case when (A, S) are reciprocally continuous semi-compatible maps and (B, T) is weakly compatible. Since A and S are reciprocally continuous semi-compatible maps, so we have $ASx_{2n} \rightarrow Au$, $SAx_{2n} \rightarrow Su$ and $M(ASx_{2n}, Su, t) = 1$. Therefore, we get $Au = Su$.
We claim that $Au = u$. For this, suppose that $Au \neq u$.
Then, setting $x = u$ and $y = x_{2n+1}$ in contractive condition (ii) with $\alpha = 1$, we get
$M(Au, Bx_{2n+1}, t) \geq r(\min\{M(Au, Su, t), M(Bx_{2n+1}, Tx_{2n+1}, t), M(Su, Tx_{2n+1}, t),$
$$M(Au, Tx_{2n+1}, t), M(Su, Bx_{2n+1}, t)\}).$$
Letting $n \rightarrow \infty$, we get $M(Au, u, t) \geq r(M(Au, u, t)) > M(Au, u, t)$, which implies that $u = Au$.
Thus, we have $u = Au = Su$. Since $AX \subseteq TX$, so there exists v in X such that $u = Au = Tv$.
Therefore, setting $x = x_{2n}$ and $y = v$ in contractive condition (ii) with $\alpha = 1$, we get
$M(Ax_{2n}, Bv, t) \geq r(\min\{M(Ax_{2n}, Sx_{2n}, t), M(Bv, Tv, t), M(Sx_{2n}, Tv, t),$
$$M(Ax_{2n}, Tv, t), M(Sx_{2n}, Bv, t)\}).$$
Letting $n \rightarrow \infty$, we get $M(Au, Bv, t) \geq r(M(Au, Bv, t)) > M(Au, Bv, t)$, which implies that $u = Bv$.
Thus, we have $u = Bv = Tv$. Therefore, we get $u = Au = Su = Bv = Tv$.

Now, since u = Bv = Tv, so by the weak compatibility of (B, T), it follows that BTv = TBv and so we get Bu = BTv = TBv = Tu. Thus, from the contractive condition (ii) with α = 1, we have
M(Au, Bu, t) ≥ r(min{M(Au, Su, t), M(Bu, Tu, t), M(Su, Tu, t),
M(Au, Tu, t), M(Su,Bu, t)}),
that is, M(u, Bu, t) > M(u, Bu, t), which is a contradiction. This implies that u = Bu. Similarly, using condition (ii) with α = 1, one can show that Au = u. Therefore, we have u = Au = Bu = Tu = Su. Hence, the point u is a common fixed point of A, B, S and T.

Again, we consider the case when (B, T) are reciprocally continuous semi-compatible maps and (A, S) is weakly compatible Since B and T are reciprocally continuous semi-compatible maps, so we have $BTx_{2n} \to$ Bu, $TBx_{2n} \to$ Tu and M(TBx_{2n}, Bu, t) = 1. Therefore, we get Bu = Tu.
We claim that Bu = u. For this, suppose that Bu ≠ u.
Then, setting $x = x_{2n}$ and $y =$ u in contractive condition (ii) with α = 1, we get
M(Ax_{2n}, Bu, t) ≥ r(min{M(Ax_{2n}, Sx_{2n}, t), M(Bu, Tu, t), M(Sx_{2n}, Tu, t),
M(Ax_{2n}, Tu, t), M(Sx_{2n}, Bu, t)}).
Letting n →∞, we get M(u, Bu, t) ≥ r(M(u, Bu, t)) > M(u, Bu, t), which implies that u = Bu.
Thus, we have u = Bu = Tu. Since BX ⊆ SX, so there exists w in X such that u = Bu = Sw.
Therefore, setting $x =$ w and $y = x_{2n+1}$ in contractive condition (ii) with α = 1, we get
M(Aw, Bx_{2n+1}, t) ≥ r(min{M(Aw, Sw, t), M(Bx_{2n+1}, Tx_{2n+1}, t), M(Sw, Tx_{2n+1}, t),
M(Aw, Tx_{2n+1}, t), M(Sw, Bx_{2n+1}, t)}).
Letting n →∞, we get M(Aw, Bu, t) ≥ r(M(Aw, Bu, t)) > M(Aw, Bu, t), which implies that u = Aw. Thus, we have u = Aw = Sw. Therefore, we have u = Aw = Sw = Bu = Tu.
Now, since u = Aw = Sw, so by the weak compatibility of (A, S), it follows that ASw = SAw and so we get Au = ASw = SAw = Su. Thus, from the contractive condition (ii) with α = 1, we have
M(Au, Bu, t) ≥ r(min{M(Au, Su, t), M(Bu, Tu, t), M(Su, Tu, t), M(Au, Tu, t), M(Su, Bu, t)}),
that is, M(Au, u, t) ≥ r(M(Au, u, t)) > M(Au, u, t), which is a contradiction. This implies that Au = u. Similarly, using (ii) with α = 1, one can show that Su = u. Therefore, we have u = Au = Bu = Tu = Su. Hence, the point u is a common fixed point of A, B, S and T.

UNIQUENESS
The uniqueness of a common fixed point of the mappings A, B, S and T be easily verified by using (ii). In fact, if u_0 be another fixed point for mappings A, B, S and T. Then, for α = 1, we have
M(u, u_0, t) = M(Au, Bu_0, t) ≥ r(min{M(Au, Su, t), M(Bu_0, Tu_0, t), M(Su, Tu_0, t),
M(Au, Tu_0, t), M(Su,Bu_0, t)}),
≥ r(M(u, u_0, t)) > M(u, u_0, t), and hence, we get u = u_0.
This completes the proof of the theorem.

We now give an example to illustrate the above theorem.
EXAMPLE: Let X = [2, 20] and M be the usual fuzzy metric space on (X, M, ∗). Define A, B, S and T : X → X as follows:
A2 = 2,   Ax = 3 if, x > 2;
Bx = 2 if, x = 2 or > 5,   Bx = 6 if, 2 < x ≤ 5;
S2 = 2,   Sx = 6 if, x > 2;
T2 = 2, Tx = 12 if, 2 < x ≤ 5, Tx = x − 5 if, x > 5.

Also, we define M(Ax, By, t) = $\dfrac{t}{t + d(x, y)}$, for all x, y in X and for all t > 0. Then, for α = 1, the pair (A, S) is reciprocally continuous semi-compatible mappings and (B, T) are weakly compatible mappings. Also, these mappings satisfy all the conditions of the above theorem and have a unique common fixed point x = 2.

REMARKS

As the earlier fixed point theorems have been established using stronger contractive conditions, so our results generalize the results of M.S. Chauhan *et. al.*[4], Singh and Jain [23], Mishra *et.al.*[15], Kutukcu *et.al.*[13] and that of Sharma [20], Mishra [14], Khan et.al.[11], Singh and Chauhan [22]. Consequently, it improves and unifies the results of Balasubramaniam *et.al.*[1], Chugh and Kumar [5], Jha [8], Jha *et al.* [9], Pant and Jha [17], Pant [18], Sharma *et al.*[21] and other similar results for fixed points in fuzzy metric space.

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