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A RESULT IN DISLOCATED QUASI METRIC SPACE

[1]D. Panthi*, [2]K. Jha

[1]Department of Mathematics, Valmeeki Campus, Nepal Sanskrit University, Nepal.
[2]School of Science, Kathmandu University, Dhulikhel, Nepal.

*Corresponding author: panthid06@gmail.com
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ABSTRACT
In this paper, we prove a fixed point theorem in dislocated quasi-metric space which extends and unifies some well-known similar results in literature.

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Keywords: fixed point, dq-metric, dq-Cauchy sequence, dq-convergent.

INTRODUCTION
In 1922, S. Banach [1] proved a fixed point theorem for contraction mapping in complete metric space. In 1986, S. G. Matthews [2] initiated the concept of dislocated metric space under the name of metric domains. In 1994, S. Abramski and A. Jung [3] presented some facts about dislocated metric in the context of domain theory. In 2000, P. Hitzler and A. K. Seda [4] generalized the celebrated Banach contraction principle in complete dislocated metric space. The notion of dislocated quasi metric space was first time introduced by F. M. Zeyada, G. H. Hassan and M. A. Ahmad [5] in 2006. It is a generalization of the result due to Hitzler and Seda in dislocated metric space. In 2008, C. T. Aage and J. N. Salunke [6] proved some results in dislocated and dislocated quasi-metric spaces. In 2010, A. Isufati [7], in 2012, K. Jha and D. Panthi [8]. The purpose of this paper is to establish a fixed point theorem for self mapping in dislocated quasi- metric space which generalizes and unifies some existing results.

We start with the following definitions.

DEFINITION 1**.** [5] Let X be a nonempty set and let $d : X \times X \to [0,\infty)$ be a function satisfying following conditions:
(i) $d(x, y) = d(y, x) = 0$, implies $x = y$, and
(ii) $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in$ X.
Then, d is called a dislocated quasi-metric (or simply dq-metric) on X.

DEFINITION 2**.** [5] A sequence $\{x_n\}$ in dislocated quasi-metric space (dq-metric space) (X, *d*) is called dq-Cauchy sequence if for given $\epsilon > 0$, there corresponds $n_0 \in N$, such that for all $m, n \ge n_0$, we have $d(x_m, x_n) < \epsilon$; or $d(x_n, x_m) < \epsilon$.

DEFINITION 3**.** [5] A sequence $\{x_n\}$ in a dq-metric space (X,d) is said to be dislocated quasi convergent (for short dq-convergent) to *x* if $lim_{n\to\infty}\ d(x_n, x) = lim_{n\to\infty}\ d(x, x_n) = 0$. In this case, *x* is called a dq-limit of $\{x_n\}$ and we write $x_n \to x$.

DEFINITION 4. [5] A dq-metric space (X, *d*) is called complete if every dq-Cauchy sequence in it is a dq- convergent.

We state the following lemmas without proof.

LEMMA 1. [5] Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0.

LEMMA 2. [5] Let (X, *d*) be a dq-metric space. If f : X→ X is a contraction function, then $\{f^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in X$.

LEMMA 3. [5] dq-limits in a dq-metric space are unique.

DEFINITION 5. [5] Let (X, *d*) be a dq-metric space. A map $T : X \to X$ is called *contraction* if there exists $0 \le \lambda < 1$ such that $d(Tx, Ty) \le \lambda\, d(x, y)$.

We present the theorems which our theorem unifies and generalizes.

In 1977, D. S. Jaggi established the following theorem in complete metric space.

THEOREM 1. [10] Let *T* be a continuous self map defined on a complete metric space (X, *d*). Further let *T* satisfies the following contractive conditions

$$d(Tx,Ty) \le \alpha \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \beta d(x,y) \qquad (1)$$

for all $x, y \in X, x \ne y$ for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$, then T has a unique fixed point.

F. M. Zeyada, G. H. Hassan and M. A. Ahmed established the following theorem in 2006.

THEOREM 2. [5] Let (X, d) be a complete dq-metric space and let $T : X \to X$ be a continuous contraction mapping, then T has a unique fixed point.

R. Shrivastava, Z. K. Ansari and M. Sharma proved the following theorems in 2012.

THEOREM 3. [11] Let T be a continuous self map defined on a complete dq- metric space (X, *d*). Further, let T satisfies the contractive condition (1) for all $x, y \in X, x \ne y$ for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$, then T has a unique fixed point.

THEOREM 4. [11] Let (X, d) be a complete dislocated quasi-metric space. Let $T : X \to X$ be continuous mapping satisfying the condition,

$$d(Tx,Ty) \le \alpha d(x,y) + \beta \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \gamma[d(x,Tx) + d(y,Ty)]$$
$$+\delta[d(x,Ty) + d(y,Tx)]$$

for all $x, y \in X$, $\alpha, \beta, \gamma, \delta > 0$, with $0 \le \alpha + \beta + 2\gamma + 2\delta < 1$, then T has a unique fixed point.

K. Zoto, E. Hoxha and A. Isufati established the following theorem in 2012.

THEOREM 5. [12] Let (X, *d*) be a complete dislocated quasi-metric space. Let $T : X \to X$ be continuous mapping satisfying the condition.

$$d(Tx,Ty) \le \alpha d(x,y) + \beta \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \gamma[d(x,Tx)+d(y,Ty)]$$
$$+\delta[d(x,Ty)+d(y,Tx)] + \eta[d(x,Tx)+d(x,y)]$$

for all $x, y \in X$, $\alpha,\beta,\gamma,\delta,\eta > 0$, with $0 \le \alpha+\beta+2\gamma+2\delta+2\eta < 1$, then T has a unique fixed point.

In 2013, D. Panthi, K. Jha and G. Porru established the following theorem.

THEOREM 6. [9] Let (X, *d*) be a complete dislocated quasi-metric space. Let $T : X \to X$ be continuous mapping satisfying the condition,

$$d(Tx,Ty) \le \alpha d(x,y) + \beta \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \gamma[d(x,Tx)+d(y,Ty)]$$
$$+\delta[d(x,Ty)+d(y,Tx)] + \eta[d(x,Tx)+d(x,y)]$$
$$+\kappa[d(y,Ty)+d(x,y)]$$

for all $x, y \in X$, $\alpha,\beta,\gamma,\delta,\eta,\kappa \ge 0$, with $0 \le \alpha+\beta+2\gamma+4\delta+2\eta+2\kappa < 1$, then T has a unique fixed point.

Now, we establish the following theorem as a main result.

MAIN RESULT

THEOREM 7. Let (X, *d*) be a complete dislocated quasi-metric space. Let $T : X \to X$ be continuous mapping satisfying the condition,

$$d(Tx,Ty) \le \alpha d(x,y) + \beta \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \gamma[d(x,Tx)+d(y,Ty)] \qquad (2)$$
$$+\delta[d(x,Ty)+d(y,Tx)] + \eta[d(x,Tx)+d(x,y)]$$
$$+\kappa[d(y,Ty)+d(x,y)] + \mu \frac{d(x,Ty).d(x,Tx)}{d(x,y)}$$

for all $x, y \in X$, $\alpha,\beta,\gamma,\delta,\eta,\kappa,\mu \ge 0$, with $0 \le \alpha+\beta+2\gamma+4\delta+2\eta+2\kappa+2\mu < 1$, then T has a unique fixed point.

PROOF:

Let us define a sequence $\{x_n\}$ as follows:

$T(x_n) = x_{n+1}$, for n = 0, 1, 2, ...

Also, let $x = x_{n-1}, and \quad y = x_n$, Then by condition (2) we have,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\le \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, Tx_{n-1}).d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}$$
$$+\gamma[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + \delta[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})$$
$$+\eta[d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)] + \kappa[d(x_n, Tx_n) + d(x_{n-1}, x_n)]$$
$$+\mu \frac{d(x_{n-1}, Tx_n).d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, x_n)}$$
$$= \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$
$$+\delta[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + \eta[d(x_{n-1}, x_n) + d(x_{n-1}, x_n)]$$
$$+\kappa[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + \mu \frac{d(x_{n-1}, x_{n+1}).d(x_{n-1}, x_n)}{d(x_{n-1}, x_n)}$$
$$\le (\alpha+\gamma+2\delta+2\eta+\kappa+\mu) d(x_{n-1}, x_n) + (\beta+\gamma+2\delta+\kappa+\mu) d(x_n, x_{n+1})$$

hence,

$$d(x_n, x_{n+1}) \le \frac{\alpha+\gamma+2\delta+2\eta+\kappa+\mu}{1-(\beta+\gamma+2\delta+\kappa+\mu)} d(x_{n-1}, x_n), \qquad 0 \le \lambda < 1$$

Thus, we have

$$d(x_n, x_{n+1}) \le \lambda d(x_{n-1}, x_n).$$

where

$$\lambda = \frac{\alpha+\gamma+2\delta+2\eta+\kappa+\mu}{1-(\beta+\gamma+2\delta+\kappa+\mu)}.$$

Similarly, we get $d(x_{n-1}, x_n) \le \lambda d(x_{n-2}, x_{n-1})$.
Hence, we have

$$d(x_n, x_{n+1}) \le \lambda^n d(x_0, x_1),$$

Now for any, m, n, $m > n$ using triangle inequality we get,

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\le \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \ldots + \lambda^{m-1} d(x_0, x_1) \\ &\le (\lambda^n + \lambda^{n+1} + \lambda^{n+2} \ldots) d(x_0, x_1) \\ &= \frac{\lambda^n}{1-\lambda} d(x_0, x_1) \end{aligned}$$

for any ε, choose $N \ge 0$ such that $\frac{\lambda^N}{1-\lambda} d(x_0, x_1) < \varepsilon$
then for any, $m > n \ge N$,
$d(x_n, x_m) \le \frac{\lambda^n}{1-\lambda} d(x_0, x_1) \le \frac{\lambda^N}{1-\lambda} d(x_0, x_1) < \varepsilon$
Similarly we can show that, $d(x_m, x_n) < \varepsilon$
Hence, $\{x_n\}$ is a Cauchy sequence in complete dislocated quasi- metric space (X, *d*). So, there exists a point $u \in X$ such that $\{x_n\} \to u$.
Since T is continuous, so, we have

$$T(u) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

UNIQUENESS:

If possible, let u and v are two fixed points of T so that, by definition, $Tu = u$ and $Tv = v$.
Let u be fixed. Then, the condition (2) gives

$$\begin{aligned} d(u,u) &= d(Tu, Tu) \\ &\le \alpha d(u,u) + \beta d(u,u) + 2\gamma d(u,u) + 2\delta d(u,u) + 2\eta d(u,u) + 2\kappa d(u,u) \\ &= (\alpha+\beta+2\gamma+2\delta+2\eta+2\kappa+\mu) d(u,u) \end{aligned}$$

which implies that $d(u,u) = 0$, since $0 \le \alpha+\beta+2\gamma+4\delta+2\eta+2\kappa+2\mu < 1$.
Thus, we have $d(u,u) = 0$
Similarly, we can get $d(v,v) = 0$, for v fixed. Again, from (2), we have

$$
\begin{aligned}
d(u,v) &= d(Tu,Tv) \\
&\le \alpha d(u,v) + \beta \frac{d(u,u).d(v,v)}{d(u,v)} + \gamma[d(u,u)+d(v,v)] \\
&+\delta[d(u,v)+d(v,u)] + \eta[d(u,u)+d(u,v)] + \kappa[d(v,v)+d(u,v)] + \mu d(u,u) \\
&= (\alpha+\delta+\eta+\kappa)d(u,v) + \delta d(v,u).
\end{aligned}
$$

Similarly, we get

$$d(v,u) \le (\alpha+\delta+\eta+\kappa)d(v,u) + \delta d(u,v).$$

Hence, we obtain

$$|\, d(u,v) - d(v,u) \,| \le (\alpha+\eta+\kappa)\,|\, d(u,v) - d(v,u) \,|,$$

which is a contradiction. So, we have $d(u,v) = d(v,u)$.
Again by (2) with substitutions we obtain, $d(u,v) \le (\alpha + 2\delta + \eta + \kappa)d(u,v)$.
which implies that, $d(u,v) = 0$. Hence, we have $d(u,v) = d(v,u) = 0$.
Therefore, we have $u = v$. This completes the proof of theorem.

REMARKS

In Main Theorem 7

1. If we put $\mu = 0$, we get the theorem 12 of D. Panthi et.al [9].
2. If we put $\kappa = \mu = 0$, we get the Theorem 3.1 of K. Zoto et.al [12].
3. If we put $\eta = \kappa = \mu = 0$, we obtain the Theorem 3.5 of R. Shrivastava et.al [11].
4. If we put $\gamma = \delta = \eta = \kappa = \mu = 0$, we obtain Theorem 3.3 of R. Shrivastava et.al [11].
5. If we put $\beta = \eta = \kappa = \mu = 0$, we obtain the Theorem 3.3 of C. T. Aage and J. N Salunke [6].
6. If we put $\delta = \eta = \kappa = 0$ then we get the theorem 3.5 of C. T. Aage and J. N. Salunke [6] with their two coefficients equal.
7. If we put $\beta = \gamma = \eta = \kappa = \mu = 0$, we get Theorem 3.2 of A. Isufati [7] with their two coefficients equal.
8. If we put $\beta = \gamma = \delta = \eta = \kappa = \mu = 0$, we get the Theorem 2.1 of F. M. Zeyada, G. H. Hassan and M. A. Ahmed [5].

Thus, our result extends and unifies the results of [5], [6], [7], [9], [11], [12] and other similar results.

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