



NON-COMPACTNESS OF A CLOSED AND BOUNDED SET

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ABSTRACT

If every closed and bounded set in a metric space is compact, the space is said to have the Heine-Borel property. This property holds in every finite dimensional normed space, but may not be true in general. Though its proof appears in many basic analysis courses, it is hard to motivate as the result is subtle and the applications are not obvious. Our goal is to provide an elegant proof as a resource for teachers that will enable them to motivate the study of this essential property and to understand the mathematics in it as a valuable teaching tool.

Keywords: Compactness, Heine-Borel property, metric space, Banach space

INTRODUCTION

The term compactness in mathematical analysis is quite familiar and occurs in many areas with plenty of applications. Having a finite sub-cover of a set A in a metric space from every open covering of A , is the compactness of the set A . Choosing the metric space simply to be R^n , a set A in R^n is compact if and only if it is closed and bounded. This is Heine-Borel theorem. The history of this theorem particularly starts in the 19th century with the exploration of solid foundation of real analysis. The theorem however does not hold as stated for general metric spaces. A metric space is said to have the Heine-Borel property if every closed and bounded subset is compact. Many metric spaces fail to have the Heine-Borel property. For example, any incomplete metric space fails to have the property. Even complete metric spaces may do the same. The Heine-Borel theorem can be generalized to arbitrary metric spaces by strengthening the conditions required for compactness: 'A subset of a metric space X is compact if and only if X is complete and totally bounded'. This article is however mainly concerned with the non-compactness of a closed and bounded set in a metric space. For this, we give an example of a non-compact set in an infinite-dimensional complete normed space though it is closed and bounded. It is an example to show that the Heine-Borel property is no longer true in an infinite dimensional Banach space. Compactness of a closed and bounded set in a finite dimensional normed space is however always preserved, the proof of which will be included as an additional part of this article. It is assumed that this article will be beneficial to the fresh master degree students who are first exposed to the study of mathematical analysis and those who may not have an access to the study of functional analysis.

**PRILIMINARIES**

Definition 2.1 (Metric Space) Let X be a set. A function $d : X \times X \rightarrow [0, \infty)$ is called a metric if $\forall x, y, z \in X$

- (i) $d(x, y) = 0 \Leftrightarrow x = y$,
- (ii) $d(x, y) = d(y, x)$ and
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$.

The set X together with the metric d is called the metric space and is denoted by (X, d) or simply by X .

For example, the Euclidean space R^n with the metric

$$d(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

and the space

$$l^2 = \{x = (x_i) = (x_1, x_2, \dots) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$$

with the metric

$$d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

are metric spaces of finite and infinite dimensions respectively. Both the above metric spaces are complete.

Definition 2.2 (Cauchy sequence) A sequence (x_n) in a metric space X is said to be Cauchy if for every $\varepsilon > 0$, there is a number $N = N(\varepsilon) > 0$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > N$.

Definition 2.3 (Convergence) A sequence (x_n) in a metric space X is said to be convergent if there is an $x \in X$ such that for every $\varepsilon > 0$, there is a number $N = N(\varepsilon) > 0$ such that $d(x_n, x) < \varepsilon$ for all $n > N$.

Note that a metric space X is complete if every Cauchy sequence in X converges. Moreover a subset M of a metric space X is compact if every sequence in M has a convergent subsequence whose limit is an element of M .

Definition 2.4 (Normed Space) A norm on a vector space X is a real valued on X whose value at $x \in X$ is denoted by $\|x\|$ and which has the properties

- (i) $\|x\| \geq 0$ (ii) $\|x\| = 0 \Leftrightarrow x = 0$
- (iii) $\|\alpha x\| = |\alpha| \|x\|$

- (iv) $\|x\| + \|y\| \leq \|x + y\|$

for all $x, y, z \in X$ and any scalar α .



The vector space X together with a norm $\| \cdot \|$ is called a normed space and is denoted by $(\| X, \| \cdot \|)$ or simply by X . Note that the spaces R^n and l^2 are finite and infinite dimensional complete normed spaces (Banach spaces) with the norms $\| x \| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$ and $\| x \| = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}$ respectively. Note that the metrics of the above spaces can be seen by the formula $d(x, y) = \| x - y \|$.

MAIN RESULTS

As a main result of this article, we show that the infinite dimensional Banach space l^2 fails to have the Heine-Borel property. Moreover the result is strengthened by adding a theorem in which the compactness of a closed and bounded set in a finite dimensional normed space is always preserved.

Theorem 3.1 *A compact subset M of a metric space X is closed and bounded, but the converse is not true in general.*

Proof. Let $x \in \overline{M}$ where \overline{M} denotes the closure of M . Then there is a sequence (x_n) in M $x_n \rightarrow x$. But M being compact, $x \in M$. So $\overline{M} = M$ and hence M is closed.

Next we show that M is bounded. Suppose that M is unbounded. Then for every positive integer n , there is an element $y_n \in M$ such that $d(b, y_n) > n$ for some fixed element $b \in M$. Clearly (y_n) is a sequence in M . Since M is compact, (y_n) has a convergent subsequence (y_{n_j}) whose limit say y is an element of M . But for $n_j > 1 + d(b, y)$, we have $d(y_{n_j}, y) \geq d(y_{n_j}, b) - d(b, y) > 1$, which is a contradiction to the fact that (y_{n_j}) converges to y . Hence M must be bounded.

It remains to show that the converse is not true in general. Let $X = l^2$ Then X is an infinite dimensional Banach space. We consider the sequence $M = \{e_n\}_{n=1}^{\infty}$ where $e_n = (\delta_{nk})$, i.e. whose n^{th} term is 1 and all other terms are 0. Then M is bounded subset of X since $\|e_n\| = 1$ for all n . Moreover we see that $d(e_i, e_j) = \left(\sum_{k=1}^{\infty} |\delta_{ik} - \delta_{jk}|^2 \right)^{\frac{1}{2}} = \sqrt{2}$ for all e_i and e_j with $i \neq j$ which follows that M is not Cauchy. Furthermore we show that M has no limit point. On the other hand suppose that it has a limit point. Then the sequence $M = \{e_n\}_{n=1}^{\infty}$ is a convergent sequence and hence it is a Cauchy, which is a contradiction. So M has no limit point and hence it is closed.

Next if $M = \{e_n\}_{n=1}^{\infty}$ were compact, the sequence $\{e_n\}_{n=1}^{\infty}$ itself must have contained a convergent subsequence, say $T = (e_{n_k})$ whose limit is an element of M . Then $T = (e_{n_k})$ is a



Cauchy sequence since every convergent sequence in a metric space is Cauchy. But this is a contradiction to the fact that $T = (e_{n_k})$ is not a Cauchy sequence because $\|x_m - x_n\| = 2$ for all $x_m, x_n \in T$ with $m \neq n$. This proves the theorem.

Theorem 3.2 *Let X be a finite dimensional normed space. Then any subset M of X is compact if and only if M is closed and bounded.*

Proof. First suppose that M is compact. Since a normed space is a metric space, the closedness and the boundedness follows from the first part of Theorem 1.

Conversely suppose that M is closed and bounded. Let dimension of M be n and $\{e_1, e_2, \dots, e_n\}$ be a basis for X . Let (x_m) be any sequence in M . Then each x_m has a representation $x_m = \alpha_1^{(m)}e_1 + \dots + \alpha_n^{(m)}e_n$. From boundedness of M , clearly (x_m) is bounded and there exists a number K such that $\|x_m\| \leq K$ for all m . Since $\{e_1, e_2, \dots, e_n\}$ is a set of linearly independent set of vectors in a normed space X , there is a number $c > 0$ such that

$$K \geq \|x_m\| = \sum_{i=1}^n \alpha_i^{(m)} e_i \geq c \sum_{i=1}^n |\alpha_i^{(m)}|.$$

It follows that the sequence of numbers $\{\alpha_i^{(m)}\}$ keeping i fixed is bounded, and by Bolzano-Weierstrass theorem, it has a limit point α_i , ($1 \leq i \leq n$). Then (x_m) has a subsequence (z_m) which converges to $z = \sum_{i=1}^n \alpha_i e_i$. Since M is closed, $z \in M$. Thus the arbitrary sequence (x_m) in M has a subsequence which converges in M , which proves the compactness of M .

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