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DEVELOPMENT OF SOME COMMON FIXED POINT THEOREMS IN SEMI-METRIC SPACE

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ABSTRACT
The purpose of this paper is to briefly study the development of some common fixed point theorems in semi-metric space.

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Keywords: Common fixed point, semi-metric space, fuzzy semi-metric space.

INTRODUCTION
Metric fixed point theory is a branch of fixed point theory which has its primary applications in functional analysis. Apart from establishing the existence of a fixed point, it often becomes necessary to prove the uniqueness of the fixed point. Besides, from computational point of view, an algorithm for calculating the value of the fixed point to a given degree of accuracy is desirable. Often this algorithm involves the iteration of the given function. In essence, the question about the existence, uniqueness and approximation of fixed point provide three significant aspect of the general fixed point principle. Among several fixed point theorems, Brouwer's fixed point theorem is well known due to its remarkable application in different fields of mathematics. The theorem is supposed to have originated from L. Brouwer's observation of a cup of coffee. If one stirs to dissolve a lump of sugar, it appears there always a point without motion. He drew the conclusion that at any moment there is a point on the surface that is not moving. The fixed point is not necessarily the point that seems to be motionless since the centre of the turbulence moves a little bit. The development of fixed point theory which is cardinal branch of non-linear analysis has given great efforts in the advancement of non-linear analysis. The earliest results had been obtained in 1920's.

In 1922, Polish mathematician Stephan Banach [4] established Banach's contraction principle (BCP) in his Ph.D. dissertation. It is also known to be Banach fixed point theorem or principle of contraction mapping. It has become milestone to all the students of mathematical analysis to establish new theorems by generalizing this theorem. The BCP has been considered to be very important as it is a source of existence and uniqueness theorem in different branches of sciences. This theorem provides an illustration of the unifying aspects in functional analysis. The important feature of the BCP is that it gives the existence, uniqueness and the sequence of the successive approximation converges to a solution of the problem.

Fixed point theory in semi-metric space is one of the emerging areas of interdisciplinary mathematical research. In 1928, K. Menger [27] introduced semi-metric space as generalization of metric space. In 1986, G. Jungck [20] introduced the notion of compatible mappings. M. Aamri and D. El Moutawakil [1] introduced the notion of property (E.A.) which is a generalization of compatible as well as non-compatible mappings. So far, various types of compatible mappings have been established by various authors. Among various type of compatible mappings, in 2007, M. R. Singh and Y. M. Singh [43] introduced the concept of compatible mappings of type (E).

The concept of fuzzy set was introduced by Iranian –American Engineer A. L. Zadeh [45] in 1965 as a new way to represent vagueness in our everyday life. Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character. But in real life situation, the problem in economics, engineering, environment, social science, medical science, etc. does not always involve crisp data.

Consequently, the last three decades were very productive for fuzzy mathematics and the recent literature has observed the fuzzification in almost every direction of mathematics such as arithmetic, topology, graph theory, probability theory, logic etc. Fuzzy set theory has application in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system). O. Kramosil and J. Michalek [23] introduced the concept of fuzzy metric space (briefly FM Space) in 1975, which opened an avenue for further development of analysis in such spaces. In 1994, A. George and P. Veeramani[8] modified the notion of fuzzy metric space with the help of continuous t-norms and have generalized several fixed point theorems. S. N. Mishra, N. Sharma and S.L. Singh [29] in 1994 introduced the notion of compatible mappings under the name of asymptotically commuting maps in FM space. B. Singh and S. Jain [42] introduced the concept of weak compatibility in fuzzy metric space in 2005. V. Pant and R.P.Pant [31] in 2007 introduced the notion of non-compatible mappings in fuzzy metric space. In 2012, T. K. Samanta et. al [38] introduced the notion of fuzzy semi-metric space and established the common fixed point theorem using various contractive conditions.

Definition 1.1 Consider a map T : $X \to X$ then any point $x \in X$ is said to be a ***fixed point*** of T if T$x = x$.

Example 1.2[33] Consider the quadratic equation $x^3 - 2x^2 - 5x + 6 = 0$, then the points x = -2, 1, 3 are the roots of this equation. This equation can be written as $x = f(x) = \frac{x^3+6}{2x+5}$. Then, it is a function equation. Since $f(-2) = -2$, $f(3) = 3$ and $f(1) = 1$, so by the definition of fixed point, the points x = -2, 1 and 3 are fixed points of f.

Definition 1.3 [7]: A ***metric space*** is a non- empty set X of objects together with a function d from $X \times X \to [0, \infty)$ satisfying the following properties for all points x, y, z in X,

M 1. $d(x, x) = 0$. (Positivity)
M 2. $d(x, y) > 0$, if $x \neq y$. (Indiscernible)
M 3. $d(x, y) = d(y, x)$ (symmetricity)
M 4. $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality)

Then d is called a metric for distance function and the pair (X, d) is called a metric space. The space is denoted simply by X if the metric is understood.

Definition 1.4 [41]: A sequence $\{x_n\}$ in a metric space (X, d) is called a ***Cauchy sequence*** if it satisfies the following condition (called the Cauchy condition): For every $\varepsilon > 0$ there is an integer N such that $d(x_n, x_m) < \varepsilon$ whenever $n \geq N$ and $m \geq N$.

Definition 1.5 [24]: A metric space (X, d) is called ***complete*** if every Cauchy sequence in X converges in X. A subset M of X is called complete if the metric subspace (M, d) is complete.

The following is the famous Banach Contraction principle introduced by S. Banach in 1922.

Theorem 1.6 [14]: Any contraction mapping T defined on a non-empty complete metric space (X, d) into itself has a unique fixed point x^* on X. Moreover, if x_o is any arbitrary point in X and the sequence $\{x_n\}$ is defined by $x_{n+1} = Tx_{n+1}$ for n = 0, 1, 2, 3 … Then $\lim_{n\to\infty} x_n = x$ and we have the estimate $d(x_n, x^*) \leq \frac{k^n}{1-k} d(x_0, x_1)$.

Rajopadhyaya & Jha Vol. 11, No. II, December, 2015, pp 68-87.

In 1928, Austrian Mathematician K. Manger introduced semi-metric space as an important generalization of metric space.

Definition 1.7 [27]: A ***semi-metric (also symmetric) space*** is a non-empty set X together with a function d: $X \times X \to [0, \infty)$ satisfying the following conditions:

SM 1. $d(x, y) = 0$ if and only if $x = y$, for $x, y \in X$, and
SM 2. $d(x, y) = d(y, x)$ for $x, y \in X$.

Example 1.8 [33] Let X = $\mathcal{R}$ be the set of all real numbers. Let a function d be defined as follows: $d(x, y) = \begin{cases} |x-y| & \text{x and y are both rational or irrational} \\ |x-y|^{-1} & \text{otherwise} \end{cases}$

Then (X, d) is a semi-metric space but not a metric space since d doesn't satisfy triangle inequality.

Example 1.9. Consider X = [0,1]. Le a function d be defined as d(x, y) = $(x-y)^2$. Then (X, d) is a semi-metric space but not a metric space since d doesn't satisfy triangle inequality.

The difference of a semi-metric space and metric space comes from the triangle inequality. In order to obtain the fixed point theorems on a semi-metric, we need some additional axioms W3, W4, W5, W, H.E. and C.C. The properties W3, W4 and W5 were introduced by W.A. Wilson [44] in 1931, H.E by M. Aamri and D. El Moutawakil [1] in 2003, W by D. Mihet [28] in 2005 and C.C by S. H. Cho, G. Y. lee and J. S. Bae [6] in 2008 as a partial replacement of triangle inequality are as follows:

W3 [44]: For a sequence $\{x_n\}$ in X, for all $x, y \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$ and $\lim_{n \to \infty} d(x_n, y) = 0$, implies that $d(x, y) = 0$ which gives $x = y$.

W4 [44]: For sequences $\{x_n\}$,$\{y_n\}$ in X and $x \in X$ $\lim_{n \to \infty} d(x_n, x) = 0$

Kathmandu University
Journal of Science, Engineering and Technology
Rajopadhyaya & Jha Vol. 11, No. II, December, 2015, pp 68-87.

and $\lim_{n \to \infty} d(y_n, x_n) = 0$, implies that $\lim_{n \to \infty} d(y_n, x) = 0$.

W5 [44]: For given sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ in X, the limits $\lim_{n \to \infty} d(y_n, x_n) = 0$

and and $\lim_{n \to \infty} d(y_n, z_n) = 0$ implies that and $\lim_{n \to \infty} d(z_n, x_n) = 0$.

H.E.[1]: For sequences $\{x_n\}$,$\{y_n\}$ in X and $x \in X$, $\lim_{n \to \infty} d(x_n, x) = 0$

and $\lim_{n \to \infty} d(y_n, x) = 0$, imply that $\lim_{n \to \infty} d(x_n, y_n) = 0$.

We have the following additional properties which are related to the continuity of semi-metric space.

C.C [6].: For a sequence $\{x_n\}$ in X, for all $x, y \in X$,

$\lim_{n \to \infty} d(x_n, x) = 0$, implies that $\lim_{n \to \infty} d(x_n, y) = d(x, y)$.

Example 1.10. Let $X = [-2,2]$ be a semi-metric space with $d(x, y) = (x - y)^2$. Consider a sequence $\{x_n\}$, $\{y_n\} \in X$ defined by $x_n = {}^1/_n + 1$ and $y_n = -{}^1/_n + 1$ which satisfies W3 and C.C properties.

W [28] : For sequences $\{x_n\}$,$\{y_n\}$ in X and $x \in X$, $\lim_{n \to \infty} d(x_n, y_n) = 0$

and $\lim_{n \to \infty} d(y_n, z_n) = 0$, implies that, $\lim_{n \to \infty} d(x_n, z_n) = 0$.

The following proposition shows the relationship between W3, W4 and C. C properties.

Proposition 1.11 [6] For axioms in semi-metric space (X, d), we have

i) W4 ⇒W and ii) C.C ⇒ W.

Definition 1.13 [3]: Let X be a nonempty set and $A, B : X \longrightarrow X$ be arbitrary mappings. A point $y \in X$ is a ***coincidence point*** for A and B if and only if $Ay = By$.

Example 1.14. Consider two self-maps A and B on $X = \mathcal{R}$ defined by $A(x) = x^2 + 1$ and $B(x) = e^x$. Then, when x = 0, A(0) = 1 and when x = 0, B(0) = 1. Also, A(0) = B(0) = 1, this imply, A(0) = B(0). Therefore, $0 \in X$ is said to be coincidence point of A and B.

Definition 1.15 [6]: let A and B be two self-mappings of a semi-metric space (X, d) then $A, B: X \longrightarrow X$ are said to be ***compatible*** if and only if $\lim_{n \longrightarrow \infty} d(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \longrightarrow \infty} d(Ax_n, t) = \lim_{n \longrightarrow \infty} d(Bx_n, t) = 0$, for some $t \in X$.

Definition 1.16 [1]: Let A and B be two self-mappings of a semi-metric space (X, d). Then A and B are said to satisfy the property ***E.A or tangential*** if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \longrightarrow \infty} d(Ax_n, t) = \lim_{n \longrightarrow \infty} d(Bx_n, t) = 0, \quad \text{for some } t \in X.$$

Definition 1.17 [25]: Let A and B be two self-mappings of a semi-metric space (X, d). Then A and B are said to be ***commuting*** if ABx = BAx for all $x \in X$. Two self-mappings A and B on a semi-metric space are said to be commuting at a point $z \in X$ if ABz = BAz.

Definition 1.18 [21]: Let A and B be two self-mappings of a semi-metric space (X, d). Then A and B are said to be ***weakly compatible*** if they commute at their coincidence points.

Definition 1.19 [22]: Let A and B be two self-mappings of a semi-metric space (X, d). Then A and B are said to be ***occasionally weakly compatible*** (owc) if there is a point $x \in X$ which is coincidence point of A and B at which A and B commute.

Example 1.20. [15] Let us consider X = [2, 20] with the semi-metric space (X, d) defined by $d(x, y) = (x - y)^2$. Define self-maps A and by

A(2) = 2 at x = 2 and A(x) = 6 for x > 2 B(2) = 2 at x = 2, B(x) = 12 for $2 < x \le 5$ and B(x) = x − 3 for x > 5. Then, for x = 9, we get A(9) = B(9) = 6. So, besides x = 2, x = 9 is another coincidence point of A and B. Also, we have AB(2) = BA(2) but AB(9) = 6, BA(9) = 3, AB(9) ≠ BA(9). Therefore A and B are occasionally weakly compatible but not weakly compatible. Hence, weakly compatible mappings are occasionally weakly compatible but not conversely in semi-metric space.

Definition 1.21 [30] Let (X, d) be a d-bounded semi-metric space and let C(X) be the set of all non-empty d-closed subset of (X, d). Consider the function $D: 2^X \times 2^X \to R^+$

defined by $D(A, B) = \max\{\sup_{a\in A} d(a, B); \sup_{b\in B} d(A, b)\}$ for $A, B \in C(X)$. Then, $(C(X), D)$ is a semi-metric space.

Definition 1.22 [33] Let X be a non-empty set and $A, B : X \to X$ be arbitrary mappings. A point $x \in X$ is a ***common fixed point for*** A and B if $Ax = Bx = x$.

Example 1.23. [33] Let $A, B: X \to X$ be functions, such that $A(x) = x^2$ and $B(x) = xe^x$. If $x = 0$, then $A(0) = 0$ and $B(0) = 0$. So, $x = 0$ is ***common fixed point*** of A and B.

Definition 1.22 [33]: Let (X, d) be a symmetric space. Then,

1) (X, d) is ***S-complete*** if for every d-Cauchy sequence $\{x_n\}$, there exists x in X such that $\lim d(x, x_n) = 0$.

2) (X, d) is ***d-Cauchy complete*** if for every d-Cauchy sequence $\{x_n\}$, there exists x in X such that $\lim x_n = x$ with respect to $t(d)$.

3) $f: X \to X$ is ***d-continuous*** if $\lim d(x_n, x) = 0$ implies $\lim d(fx_n, fx) = 0$.

4) $f: X \to X$ is ***t(d)-continuous*** if $\lim x_n = x$ with respect to $t(d)$ implies $\lim fx_n = f(x)$ with respect to $t(d)$.

Definition 1.23 [18] Let A and B be two self-mappings of a semi-metric

space (X, d). Then A and B are said to be ***compatible mapping of type (E)***

if $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ABx_n = B(t)$ and $\lim_{n\to\infty} BBx_n = \lim_{n\to\infty} BAx_n = A(t)$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Ax_n, t) = \lim_{n\to\infty} d(Bx_n, t) = 0$, for some $t \in X$.

Proposition 1.24 [18]:

Let A and B be two compatible mappings of type (E). If one of the function is continuous, then

Kathmandu University
Journal of Science, Engineering and Technology
Rajopadhyaya & Jha Vol. 11, No. II, December, 2015, pp 68-87.

i) $A(t) = B(t)$ and $\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} BBx_n = \lim_{n\to\infty} ABx_n = \lim_{n\to\infty} BAx_n$,

where $\lim_{n\to\infty} Ax_n = t$ and $\lim_{n\to\infty} Bx_n = t$.

ii) If there exists $u \in X$ such that $Au = Bu = t$, then $ABu = BAu$.

In order to establish our result, we need a function $\emptyset: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $0 < \emptyset(t) < t,\ t > 0$.

Definition 1.25 [38] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a ***continuous t-norm*** if $*$ satisfies the following conditions:

i) $*$ is commutative and associative

ii) $*$ is continuous

iii) $a * 1 = a$ for all $a \in [0,1]$

iv) $a * b \le c * d$, whenever $a \le c$, $b \le d$ and $a, b, c, d \in [0,1]$.

Example 1.26 a*b = min(a, b) is a continuous t-norm for all a , b ∈ [0,1] where $* : [0, 1] \times [0, 1] \to [0, 1]$ is a binary operation.

Definition 1.27 [8] The 3-tuple $(X, M,*)$ is called ***fuzzy metric space*** if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$ which satisfying the following conditions:
i) $M(x, y, t) > 0$
ii) $M(x, y, t) = 1$ if and only if $x = y$
iii) $M(x, y, t) = M(y, x, t)$
iv) $M(x, y, s) * M(y, z, t) \le M(x, z, s + t)$
v) $M(x, y, \cdot): (0, \infty) \to (0,1]$ is continuous, for all $x, y, z \in X$ and $t, s > 0$.

Definition 1.28 [40] The (X, M) is called a ***fuzzy semi-metric space*** if $X^2 \times (0, \infty)$ satisfying the following conditions:
i) $M(x, y, t) > 0$
ii) $M(x, y, t) = 1$ if and only if $x = y$
iii) $M(x, y, t) = M(y, x, t)$

Remark 1.32 [40] Every fuzzy metric space is a fuzzy semi-metric space but the converse is not necessarily true.

Example 1.33 [38] Consider $X = (0, \infty)$ and $M(x, y, t) = \frac{t}{t + |x - y|}$ if $x \neq 0, y \neq 0$ and $M(x, y, t) = \frac{t}{t + \frac{1}{x}}$ if $x \neq 0$. (X, M) is a fuzzy semi-metric space. Let $x = 1, y = \frac{1}{2}, z = 0$, $s = 1, t = 0$ *and* $a * b = \max\{a, b\}$. Then condition (iv) of definition 1.2 is not satisfied and hence (X, M) is not a fuzzy metric space but it is fuzzy semi-metric space.

We have following useful conditions W3, W4, H.E, introduced by Samanta and Mohinta [40] to establish fixed point results in fuzzy semi-metric space replacing triangle inequality.

($W3$)[40] For a sequence $\{x_n\}$ in, $x, y \in X$ $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} M(x_n, y, t) = 1$ imply $x = y$.

($W4$) [40] For a sequences $\{x_n\}$ and $\{y_n\}$ in, $x, y \in X$ $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} M(x_n, y_n, t) = 1$ imply that $\lim_{n\to\infty} M(y_n, x, t) = 1$.

(H.E) [40] For a sequences $\{x_n\}$ and $\{y_n\}$ in, $x \in X$ $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} M(y_n, x, t) = 1$ imply that $\lim_{n\to\infty} M(x_n, y_n, t) = 1$.

Proposition 1.34 [40] For axioms in fuzzy semi-metric space (X, M), W4 implies W3. But the converse is not true.

Definition 1.35 [40] Let A and B be two self-mappings of a fuzzy semi-metric space (X, M). Then

A pair of self-mappings A and B satisfy ***the property E.A*** if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} M(Ax_n, r, t) = \lim_{n\to\infty} M(Bx_n, r, t) = 1$.

Definition 1.36 [40] Let A and B be two self-mappings of a fuzzy semi-metric space (X, M). Then Self mappings A and B are said to be ***weakly compatible*** if they commute at their coincidence points, that is $Az = Bz$ implies that $ABz = BAz$.

2. SOME COMMON FIXED POINT RESULTS IN FUZZY/SEMI-METRIC SPACE

In this section, we state without proof some fixed point theorem in semi-metric space which are the sources for our results.

Theorem 2.1 [30] Let (X, d) be a d-bounded and S-complete semi-metric space satisfying W4 and $A: X \to C(X)$ be a multivalued mappings such that $d(Ax, By) \leq k\, d(x, y),\ k \in [0,1]$, for all $x, y \in X$. Then, there exists $u \in X$ such that $u \in Au$.

In 1999, T. L. Hicks and B. E. Rhoades established the following common fixed point theorem as an extension of Banach contraction principle in semi-metric space for pair of self-maps.

Theorem 2.2 [9] Let d be a bounded symmetric (semi-metric) for X that satisfies (W3).Suppose (X, d) is S-complete (d-Cauchy complete) and $A: X \to X$ is d-continuous ($t(d)$-continuous). Then A has a fixed point if and only if there exists $\alpha \in (0, 1)$ and a d-continuous $(t(d) -$ continuous) function $B: X \to X$ which commutes with f and satisfies

i) $B(X) \subset A(X)$ and (ii) $d(Bx, By) \leq \alpha d(Ax, Ay)$ for all $x, y \in X$.

Then, A and B have a unique common fixed point.

In particular if $A = B$ and $B = I$ an identity mapping in the above theorem, then Banach contraction principle in semi-metric space reduces to BCP in usual metric space. Now, we consider a function $\emptyset: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $0 < \emptyset(t) < t,\ t > 0$.

In2002, M. Aamri and D. El Moutawakil established the following theorem for pair of self-mappings in semi-metric space using weakly compatible self-mappings.

Theorem 2.3 [1] Let (X, d) be a semi-metric (symmetric) space which satisfies properties W4 and H.E. Let $A, B : X \longrightarrow X$ be self- mappings of X such that:

i) $AX \subseteq TX$

ii) A and B satisfy the property (E.A.);

iii) A and B are weakly compatible; and

iv) $d(Ax, Ay) \leq \varphi$ (max$\{d(Bx, By), d(Bx, Ay), d(Ay, By)\}$ for any $x, y \in X$.

If the range of one of the mappings A or B is a complete subspace of X, then A and B have a unique common fixed point.

Kathmandu University
Journal of Science, Engineering and Technology
Rajopadhyaya & Jha Vol. 11, No. II, December, 2015, pp 68-87.

Let Φ denotes the set of all real functions ϕ: $[0, \infty) \longrightarrow [0, \infty)$ with the following properties:

i) $\phi(0) = 0$,

ii) $\phi(r) <$ for all $r > 0$ and

iii) $\lim_{t \longrightarrow r} + \phi(t) < r$ for any $r > 0$.

Also, δ denotes the set of all continuous, monotone non-decreasing, real functions
F: $[0, \infty] \longrightarrow [0, \infty]$ such that $F(x) = 0$ if and only if $x = 0$.

In 2009, I. D. Randelovic and D. S. Petkovic established the following common fixed point theorem in semi-metric space using weakly compatible mappings for pair of self-mappings.

Theorem 2.4 [3] Let (X, d) be a semi-metric (symmetric) space which satisfies properties (W3) and (H.E.). Let $\phi \in \Phi$, $F \in \delta$ and let $A, B : X \rightarrow X$ be self –mappings of X such that :

(i) $F(d(Ax, Ay) \leq \varphi\ (F(\max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\}))$ for any $x, y \in X$. ;

(ii) A and B satisfy the property (E.A.);

(iii) A and B are weakly compatible; and

(iv) $AX \subseteq BX$

If the range of one of the mappings A or B is a complete subspace of X, then A and B have a unique common fixed point.

In 2002, M. Aamri and D. El Moutawakil established the following common fixed point theorem for two pairs of self-mappings in semi-metric space.

Theorem 2.5 [1] Let (X, d) be a semi-metric (symmetric) space which satisfies properties (W4) and (H.E.). Let $\varphi \in \Phi$ and let $A, B, S, T : X \longrightarrow X$ be self- mappings of X such that:

i) $d(Ax, By) \leq \varphi\ (\max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\}$ for any $x, y \in X$.

ii) (A, S) or (B, T) satisfies the property (E.A.);

Rajopadhyaya & Jha Vol. 11, No. II, December, 2015, pp 68-87.

iii) (A, S) and (B, T) are weakly compatible; and

iv) $AX \subseteq TX$ and $\subseteq SX$;

If the range of one of the mappings *A*, *B*, *S* or *T* is a complete subspace of *X*, then *A*, *B*, *S* and *T* have a unique common fixed point.

In 2009, I. D. Arandelovic and D. S. Petkovic established the following common fixed point theorem in semi-metric space for two pairs of self-mappings using W3 and H.E properties.

Theorem 2.6 [3] Let (X, d) be a semi-metric (symmetric) space which satisfies properties (W4) and (H.E.). Let $\varphi \in \Phi$, $F \in \delta$ and let $A, B, S, T : X \to X$ be self –mappings of *X* such that :

i) $F(d(Ax, By) \leq \varphi\ (F(\max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\}))$, for any $x, y \in X$.

ii) (A, S) or (B, T) satisfy the property (E.A.);

iii) (A, T) and (B, S) are weakly compatible; and

iv) $AX \subseteq TX$ and $BX \subseteq SX$;

If the range of one of the mappings *A*, *B*, *S* or *T* is a complete subspace of X, then *A*, *B*, *S* and *T* have a unique common fixed point.

In 2008, S. H. Cho, G. Y. Lee and J. S. Bae established the following common fixed point theorem in semi-metric space for two pairs of self- mappings.

Theorem 2.7 [6]: Let (X, d) be a symmetric (semi-metric) space that satisfies (W3) and (H.E.) and let A, B, S and T be self- mappings of *X* such that

i) $AX \subset TX$ and $BX \subset SX$,

ii) the pair (B, T) or (A, S) satisfies property (E.A.)

iii) the pair (B, T) and (A, S) are weakly compatible,

iii) for any $x, y \in X$, $d(Ax, By) \le m(x, y)$, where

$m(x,y) = \max\{d(Sx,Ty), \min\{d(Ax,Sx), d(By,Ty)\}, \min\{d(Ax,Ty), d(By,sx)\}\}$, and

iv) SX and TX are d-closed subset of X .

Then, A, B, S and T have a unique common fixed point in X.

In 2009, M. Imdad and J. Ali established the following common fixed point theorem in semi-metric space for two pairs of self-mappings using implicit relation.

Theorem 2.8 [11]: Let A, B, S and T be self mappings of a symmetric space (X, d) such that (i) the pair (A, S) and (B, T) satisfy the common property (E.A.),

(ii) $S(X)$ and $T(X)$are closed subsets of X,

(iii) for all $x \neq y \in X$ and $F \in \Phi$,

$F(d(Ax,By), d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Sx,By), d(Ty,Ax) < 0.$

Then, the pairs (A, S) and (B, T) have a point of coincidence. Moreover, if the pairs (A, S) and (B, T) are weakly compatible then A, B, S and T have a unique common fixed point in X.

In 2010, A. H. Soliman and M. Imdad established the following common fixed point theorem in semi-metric space for two pairs of self-mappings using S-continuous and T-continuous.

Theorem 2.9 [12] Let Y be an arbitrary non-empty set whereas X be another non-empty set with semi-metric space (X, d) which satisfies W3 and H.E. Let A, B, S, T : Y → X be four mappings which satisfies the following conditions

i) A is S-continuous and B is T-continuous.

ii) the pair (A, S) and (B, T) satisfy the common property E. A,

iii) Sx and Tx are d-closed subset of X, then there exists u, w ∈ X such that Au = Su = Tw = Bw. Moreover, if Y= X along with

iv) the pairs (A, S) and (B, T) are weakly compatible and

v) d(Ax, BAx) ≠ max{d(Sx,TAx), d(BAx, TAx), d(Ax, TAx), d(Ax, Sx), d(BAx, Sx)} whenever the right hand side is non-zero. Then A, B, S and T have a common fixed point in X.

In 2011 H. K. Pathak and R. K. Verma established the following common fixed point theorem in semi- metric space for two pairs of self-mappings using occasionally converse commuting.

Theorem 2.10 [32] Let A, B, S and T be self-mappings of a semi-metric space (X, d) satisfying

Φ(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, Sx), d(Ax, Ty) > 0, where x, y ∈ X and $\Phi \in F_6$. If one of the following conditions hold.

i) the pairs (A, S) is occasionally converse commuting and the pairs (B,T) is occasionally weakly compatible or

ii) the pairs (B, T) is occasionally converse commuting and the pairs (A, S) is occasionally weakly compatible, then A, B, S and T have a unique common fixed point in X.

In 2014, K. Jha, M. Imdad and U. Rajopadhayay [19] established the following common fixed point theorem for three pairs of mappings using weakly compatible and E. A property, that extends the result of M. Aamri and D. El Moutawakil [1]

Theorem 2.11 [19] Let (X, d) be a semi-metric space that satisfies (W4) and (HE) . Let A, B, T, S, P and Q be self- mappings of X such that

i) $ABX \subset PX$ and $TSX \subset QX$

ii) $d(ABx, TSy) \leq \emptyset(\max\{d(Qx, Py), d(Qx, TSy), d(Py, TSy)\})$ for all $(x, y) \in X \times X$

iii) (AB, Q) or (TS, P) satisfies the property E.A. , and

iv) (AB, Q) and (TS, P) are weakly compatibles.

If the range of the one of the mapping AB, TS, P and Q is a complete subspace of X then AB, TS, P and Q have a unique common fixed point. Furthermore if the pairs (A, B), (A, P), (B, P),(S,T), (S, J) and (T, Q) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

Rajopadhyaya & Jha Vol. 11, No. II, December, 2015, pp 68-87.

In 2014, K. Jha, M. Imdad and U. Rajopadhyaya [17] established the following common fixed point theorem for six mappings using weakly compatible and the property E. A that extends the results of M. Aamri and D. El Moutawakil [1] by using the property C. C and H. E only under different contraction.

We denote Λ by the class of non-decreasing continuous function $\alpha: R^+ \rightarrow R^+$ such that $(\alpha 1)$ $\alpha(0) = 0$ and $(\alpha 2)$ $\alpha(S) > 0$ for all S > 0.

Theorem 2.12 [17] Let (X, d) be a semi-metric space that satisfies (H.E.) and (C.C.). Let *A*, *B*, *T*, *S*, *P* and *Q* be self- mappings of *X* such that

(i) $ABX \subset QX$ and $\subset PX$,

(ii) $\alpha(d(ABx, TSy)) \leq \emptyset(\alpha(\max\{d(Px, Qy), d(ABx, Px), d(TSy, Qy), d(ABx, Qy), d(TSy, Px)\}))$ for all $(x, y) \in X \times X$,

(iii) the pair (TS, Q) satisfies the property E.A ,

(iv) (AB, P) and (TS, Q) are weakly compatibles, and

(v) PX is d-closed subset of *X*.

then *AB*, *TS*, *P* and Q have a unique common fixed point. Furthermore, if the pairs (*A*, *B*), (*A*, *P*), (*B*, *P*), (*S*, *T*), (*S*, *Q*) and (*T*, Q) are commuting pair of mappings then *A*, *B*, *T*, *S*, *P* and Q have a unique common fixed point.

In 2015, U. Rajopadhyaya, K. Jha and R. P. Pant [37] established the following common fixed point theorem for three pairs of mappings using weakly compatible and E.A property that extends the result of M. Imdad and J. Ali [11]

Theorem 2.13 [37] Let *A*, *B*, *T*, *S*, *P* and Q be self- mappings of semi-metric space (X, d) such that

i) $ABX \subset QX$ and $TSX \subset PX$

ii) $F(d(ABx, TSy), d(Px, Qy), d(ABx, Px), d(TSy, Qy), d(Px, TSy), d(Qy, ABx)) < 0$. Suppose that

iii) the pairs (AB, P) and (TS, Q) satisfy the property (E.A).

iv) $P(X)$ and $Q(X)$ are closed subset of X

v) the pairs (AB, P) and (TS, Q) are weakly compatible mappings. Then *AB*, *TS*, *P* and Q have a unique common fixed point. Furthermore, if the pairs (*A*, *B*) and (*T*, *S*) are commuting pair of mappings then *A*, *B*, *T*, *S*, *P* and Q have a unique common fixed point.

In 2014, U. Rajopadhyaya, K. Jha and M. Imdad [34] established the following common fixed point theorem for two pairs of mappings using compatible mappings of type (E).

Theorem 2.14 [34] Let (X, d) be a semi-metric space that satisfies (W4) and (H.E). Let A,B,T and S be self-mappings of X, such that

i) $AX \subset TX$ and $BX \subset SX$,

$ii)$ $d(Ax, By) \leq \emptyset(\max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\})$ for all $(x, y) \in X \times X$,

iii) The pair (B, T) or (A, S) satisfies E.A. property,

iv) The pair (B, T) and (A, S) are compatible mapping of type (E), and

v) SX or TX is a d-closed subset of X.

If one of the mapping A,B,T and S is continuous then A,B,T and S have a unique common fixed point.

In 2014, K. Jha, M. Imdad and U. Rajopadhyaya [15] established the following common fixed point theorem in semi-metric space for three pairs of mappings using occasionally weakly compatible mapping.

Theorem 2.15 [15] Let (X, d) be a semi-metric space. Let A, B, T, S, P and Q be self- mappings of X such that

(i) $\{AB, P\}$ and $\{TS, Q\}$ are occasionally weakly compatible (OWC),

(ii) $d(ABx, TSy) \leq \emptyset(max\{d(Px, Qy), \frac{1}{2}[d(ABx, Px) + d(TSy, Qy)], d(ABx, Qy), d(TSy, Px)\})$ for all $(x, y) \in X \times X$,

then AB, TS, P and Q have a unique common fixed point. Furthermore, if the pairs (A, B) and (T, S) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

In 2014, U. Rajopadhyaya, K. Jha and Y. J. Cho [35] established the following common fixed point theorem in semi-metric space for three pairs of mappings using occasionally converse commuting mapping.

Theorem 2.16 [35] Let (X, d) be a semi-metric space. Let A, B, T, S, P and Q be self- mappings of X such that

(i) $\{AB, P\}$ and $\{TS, Q\}$ are occasionally converse commuting (occ) and

(ii) $d(ABx,\ TSy)\ \leq\ \emptyset(max\{d(Px,Qy),\frac{1}{2}[d(ABx,Px)+d(TSy,Qy)],\frac{1}{2}[d(ABx,Qy)+d(TSy,Px)]\}$ for all $(x,y)\in X\times X$,

then AB, TS, P and Q have a unique common fixed point. Furthermore, if the pairs (A, B) and (T, S) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

In 2014, U. Rajopadhyaya, K. Jha and P. Kumam [36] established the common fixed point theorem in fuzzy semi-metric space for three pairs of mappings using weakly compatible mappings.

Theorem 2.17 [36] Let (X,M) be a fuzzy semi-metric space that satisfies (W4) and (H.E) . Let A, B, T, S, P and Q be self- mappings of X such that

i) $ABX\subset PX$ and $TSX\subset QX$,

ii) $M(ABx, TSy, t) \geq \emptyset(\min\{M(\mathrm{Qx},\mathrm{Py},\mathrm{t}),M(Qx,TSy,t),M(Py,TSy,t)\})$ for all $(x,y)\in X\times X$

iii) (AB,Q) or (TS,P) satisfies the property E.A. , and

iv) (AB,Q) and (TS,P) are weakly compatible.

If the range of the one of the mapping AB, TS, P and Q is a complete subspace of X then AB, TS, P and Q have a unique common fixed point. Furthermore if the pairs (A, B), (A, P), (B, P), (S,T), (S, J) and (T, Q) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

CONCLUSION

We have presented the development of common fixed point theorem in semi-metric space with a pair of mapping, two pair of mappings and three pair of mappings using various contractive conditions.

Future aspects of fixed point theorems in semi-metric space is as follows:

(1) Fixed point theorems in semi-metric space is an open wide area of research activities for the establishment of fixed point theorems using various compatible mappings.

(2) There is a wide scope to study common fixed point theorems in semi-metric space for sequence of mappings.

(3) Establishment of common fixed point theorems in semi-metric space using various contractive conditions.

(4) Connection of fixed point theorem in semi-metric space to economics.

(5) To find applications of fixed points in different fields.

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