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ON TRANS-SASAKIAN MANIFOLDS

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ABSTRACT

In this paper we study the geometry of trans-Sasakian manifold when it is projective Ricci-semi-symmetric, pseudo-projectively flat and pseudo-projectively semi-symmetric.

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Keywords: Trans-Sasakian manifold, projective Ricci tensor, pseudo-projective curvature tensor, pseudo-projectively flat, pseudo-projectively semi-symmetric.

1. INTRODUCTION

In 1985, J. A. Oubina [9] introduced the notion of trans-Sasakian manifold. Many geometers studied this manifold some of them are [9, 7, 1]. Semi-symmetric manifold is studied by author [10], [11] and others. The conditions $R(X,Y).\tilde{P}=0,\ \ \bar{P}(X,Y)Z=0$ and $R(X,Y).\bar{P}=0$ are called projective Ricci-semi-symmetric, pseudo-projectively flat and pseudo-projectively semi-symmetric respectively, where $R(X,Y)$ is considered as derivation of tensor algebra at each point of the manifold.

We note that trans-Sasakian structure of type $(0,0),(0,\beta)$ and $(\alpha,0)$ are called cosympletic, $\beta-$Kenmotsu and $\alpha-$Sasakian manifold respectively. Thus trans-Sasakian structures are also provide a large class of generalized quasi-Sasakian structures.

An almost contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be trans-Sasakian manifold [1] if $(M\times\mathbb{R},J,G)$ belongs to the class ω_4 [8] of the Hermitian manifolds where J is the almost complex structure on $M\times\mathbb{R}$ defined by

$$J\left(X,f\frac{d}{dt}\right)=\left(\phi X-f\xi,\eta(X)f\frac{d}{dt}\right), \tag{1.1}$$

for all vector fields on M and smooth function f on $M\times\mathbb{R}$ and G is the product metric on $M\times\mathbb{R}$. This may be stated by the condition [4]

$$(\nabla_X\phi)(Y)=\alpha\{g(X,Y)\xi-\eta(Y)X\}+\beta\{g(\phi X,Y)\xi-\eta(Y)\phi X\}, \tag{1.2}$$

for some smooth functions α and β on M and we say that trans-Sasakian structure is of type (α,β).

In this paper we consider the trans-Sasakian manifold under the condition $\phi(grad\alpha)=(2n-1)grad\beta$ satisfying $R(X,Y).\tilde{P}=0,$ $\bar{P}(X,Y)Z=0$ and $R(X,Y).\bar{P}=0,$ where $\tilde{P}$ is the projective Ricci tensor introduced by the authors [6]. It is defined by

(1.3) $$\tilde{P}(X,Y)=\frac{(2n+1)}{2n}S(X,Y)-\frac{r}{2n}g(X,Y),$$

where S and r are Ricci tensor and scalar curvature respectively. It is shown that in first condition the manifold is Einstein and its scalar curvature is $2n(2n+1)(\alpha^2-\beta^2)$.

Further, trans-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ with $\bar{P}(X,Y)Z=0$ and $R(X,Y).\bar{P}=0$, is considered, where $\bar{P}$ is the pseudo-projective curvature tensor given by [2]

(1.4) $$\bar{P}(X,Y)Z=aR(X,Y)Z+b\left[S(Y,Z)X-S(X,Z)Y\right] -\frac{r}{(2n+1)}\left\{\frac{a}{2n}+b\right\}\left[g(Y,Z)X-g(X,Z)Y\right],$$

where a,b are constants such that $a,b\neq 0, R,S,r$ are the curvature tensor, Ricci tensor and scalar curvature respectively.

2. PRELIMINARIES

Let M be a $(2n+1)-$dimensional almost contact metric manifold [3] with an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is the associated Riemannian metric such that

(2.1) $$\phi^2(X)=-X+\eta(X)\xi,$$

(2.2) $$\eta(\xi)=g(\xi,\xi)=1,\ \phi\xi=0,$$

(2.3) $$\eta(\phi X)=0,\quad \eta\circ\phi=0,$$

(2.4) $$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\ g(X,\xi)=\eta(X),$$

(2.5) $$d\eta(X,Y)=g(X,\phi Y)=-g(\phi X,Y),\ \text{for all } X,Y\in TM.$$

From (1.2) it follows that

(2.6) $$\nabla_X\xi=-\alpha\phi X+\beta\{X-\eta(X)\xi\},$$

(2.7) $$(\nabla_X\eta)(Y)=-\alpha g(\phi X,Y)+\beta g(\phi X,\phi Y).$$

Further, on a trans-Sasakian manifold the following relations hold [7], [5]:

(2.8) $$R(X,Y)\xi=(\alpha^2-\beta^2)\left[\eta(Y)X-\eta(X)Y\right]-(X\alpha)\phi Y-(X\beta)\phi^2Y +2\alpha\beta\left[\eta(Y)\phi X-\eta(X)\phi Y\right]+(Y\alpha)\phi X+(Y\beta)\phi^2X,$$

(2.9) $$R(\xi,X)\xi=(\alpha^2-\beta^2-\xi\beta)\left[\eta(X)\xi-X\right],$$

(2.10) $$2\alpha\beta+(\xi\alpha)=0,$$

(2.11) $$S(X,\xi)=\left[2n(\alpha^2-\beta^2)-(\xi\beta)\right]\eta(X)-(\phi X)\alpha-(2n-1)(X\beta),$$

(2.12) $$Q\xi=\left[2n(\alpha^2-\beta^2)-(\xi\beta)\right]\xi+\phi(grad\alpha)-(2n-1)grad\beta,$$

when $\phi(grad\alpha)=(2n-1)grad\beta,$ then the relations (2.11) and (2.12) reduce to

(2.13) $$S(X,\xi)=2n(\alpha^2-\beta^2)\eta(X),$$

(2.14) $$Q\xi=2n(\alpha^2-\beta^2)\xi,$$

(2.15) $$S(\xi,\xi)=2n(\alpha^2-\beta^2).$$

3. RESULTS AND DISCUSSION

Theorem 3.1: If in a trans-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$, the relation $R(X,Y).\tilde{P}=0$ holds, then the manifold is Einstein.

Proof: Consider a trans-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ which satisfies the condition

(3.1) $$R(X,Y).\tilde{P}=0$$

where $\tilde{P}$ is the projective Ricci Tensor defined in (1.3). Now,

(3.2) $$\left(R(X,Y).\tilde{P}\right)(U,V)=-\tilde{P}\left(R(X,Y)U,V\right)-\tilde{P}\left(U,R(X,Y)V\right).$$

From (3.1) and (3.2), we get

(3.3) $$\tilde{P}\left(R(X,Y)U,V\right)+\tilde{P}\left(U,R(X,Y)V\right)=0.$$

Putting $X=\xi$ and using (2.8) in (3.3) we have

(3.4) $$\begin{aligned}&(\alpha^2-\beta^2)\left[g(Y,U)\tilde{P}(\xi,V)-\eta(U)\tilde{P}(Y,U)+g(Y,U)\tilde{P}(\xi,U)-\eta(V)\tilde{P}(U,Y)\right]\\&-2\alpha\beta\left[\eta(U)\tilde{P}(\phi Y,V)+\eta(V)\tilde{P}(\phi Y,U)\right]-(\xi\alpha)\left[\tilde{P}(\phi Y,V)+\tilde{P}(\phi Y,U)\right]=0\end{aligned}$$

Putting $V=U$ in (3.4), we get

(3.5) $$(\alpha^2-\beta^2)\left[g(Y,U)\tilde{P}(\xi,U)-\eta(U)\tilde{P}(Y,U)\right]-\tilde{P}(\phi Y,U)\left[2\alpha\beta\eta(U)+(\xi\alpha)\right]=0.$$

Under condition $2\alpha\beta\eta(U)+\xi\alpha=0$ if $\eta(U)=1,$ using (3) and (2.13) in (3.5), we get

(3.6) $$S(U,Y)=2n(\alpha^2-\beta^2)g(U,Y).$$

This implies that the manifold is an Einstein manifold. This completes the proof of the theorem.

Let $\{e_i : i=1,2,....,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $U=Y=e_i$ in (3.6) and taking summation over i, $1\le i\le 2n+1,$ we get

(3.7) $$r=2n(2n+1)(\alpha^2-\beta^2).$$

Hence we can state:

Corollary 3.1: A projective Ricci-semi-symmetric trans-Sasakian manfold $M^{2n+1}(\phi,\xi,\eta,g),$ is the manifold of constant scalar curvature $2n(2n+1)(\alpha^2-\beta^2).$

Theorem 3.2: A pseudo-projectively flat trans-Sasakian manfold $M^{2n+1}(\phi,\xi,\eta,g)$ is an $\eta-$Einsten manifold provided that $a,b\neq 0.$

Proof: The pseudo-projective curvature tensor is given by the relation (4). Suppose $\bar{P}(X,Y)Z=0,$ then from (1.4), we get

(3.2.1) $$\begin{aligned}&aR(X,Y)Z+b\left[S(Y,Z)X-S(X,Z)Y\right]\\&-\frac{r}{(2n+1)}\left\{\frac{a}{2n}+b\right\}\left[g(Y,Z)X-g(X,Z)Y\right]=0\end{aligned}$$

Taking inner product on both sides of (3.2.1) by $\xi,$ we get

(3.2.2)
$$a\eta\big(R(X,Y)Z\big)+b\big[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\big] -\frac{r(a+2nb)}{2n(2n+1)}\big[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\big]=0$$

Putting $X=\xi$ and using (2.4),(2.8) and (2.13), in (3.2.2), we get

$$a\Big[\big(\alpha^2-\beta^2\big)\big\{g(Y,Z)-\eta(Y)\eta(Z)\big\}\Big]+b\Big[S(Y,Z)-2n\big(\alpha^2-\beta^2\big)\eta(Y)\eta(Z)\Big] -\left\{\frac{r(a+2nb)}{2n(2n+1)}\right\}\big[g(Y,Z)-\eta(Y)\eta(Z)\big]=0$$

which yields on further calculation

(3.2.3)
$$S(Y,Z)=\left[\frac{1}{b}\left\{\frac{(a+2nb)r}{2n(2n+1)}-a\big(\alpha^2-\beta^2\big)\right\}\right]g(Y,Z) +\left[\frac{(a+2nb)}{b}\left\{\big(\alpha^2-\beta^2\big)-\frac{r}{2n(2n+1)}\right\}\right]\eta(Y)\eta(Z).$$

Thus the theorem is proved.

Let $\{e_i : i=1,2,....,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $Y=Z=e_i$ in (3.2.3) and taking summation over i, $1\le i\le 2n+1,$ we get

(3.2.4)
$$r=2n(2n+1)\big(\alpha^2-\beta^2\big).$$

Hence we can state:

Corollary 3.2: A pseudo-projectively flat trans-Sasakian manifold of dimension $(2n+1)$ is of manifold of constant scalar curvature $2n(2n+1)\big(\alpha^2-\beta^2\big)$.

Using the relation (3.2.4) in (3.2.3), we get

(3.2.5)
$$S(Y,Z)=2n\big(\alpha^2-\beta^2\big)g(Y,Z).$$

This leads to the following:

Theorem 3.3: If a trans-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is pseudo-projectively flat then it is Einstein one & its scalar curvature is given by (3.2.4).

Theorem 3.4: A pseudo-projectively semi-symmetric trans-Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is an $\eta-$Einstein manifold.

Proof: Let us suppose that a $(2n+1)$-dimensional trans-Sasakian manifold satisfies the condition

(3.4.1)
$$\mathrm{R}(\mathrm{X},\mathrm{Y}).\overline{\mathrm{P}}=0.$$

where $\overline{\mathrm{P}}$ is the pseudo-projective curvature tensor given in (1.4).

Using (2.4) and (2.8) in (1.4), we get

(3.4.2) $$\eta\left(\bar{P}(U,V)W\right)=\left\{a\left(\alpha^2-\beta^2\right)-\frac{r(a+2nb)}{2n(2n+1)}\right\}\left[g(U,V)\eta(U)-g(U,W)\eta(V)\right]$$
$$+b\left[S(V,W)\eta(U)-S(U,W)\eta(V)\right].$$

Taking $U=\xi$ in (3.4.2) and using (2.2), (2.4) and (2.13), we get

(3.4.3) $$\eta\left(\bar{P}(\xi,V)W\right)=bS(V,W)\left\{a\left(\alpha^2-\beta^2\right)-\frac{(a+2nb)r}{2n(2n+1)}\right\}g(V,W)$$
$$+\left[(a+2nb)\left\{\frac{r}{2n(2n+1)}-\left(\alpha^2-\beta^2\right)\right\}\right]\eta(V)\eta(W).$$

Putting $W=\xi$ in (3.4.2) and using (2.8) and (2.13), we obtain

(3.4.4) $$\eta\left(\bar{P}(U,V)\xi\right)=0.$$

Now,

(3.4.5) $$\left(R(X,Y).\bar{P}\right)(U,V)W=R(X,Y)\bar{P}(U,V)W-\bar{P}\left(R(X,Y)U,V\right)W$$
$$-\bar{P}\left(U,R(X,Y)V\right)W-\bar{P}(U,V)R(X,Y)W.$$

From the relations (5.1) and (5.5), we have

(3.4.6) $$R(X,Y)\bar{P}(U,V)W-\bar{P}\left(R(X,Y)U,V\right)W-\bar{P}\left(U,R(X,Y)V\right)W$$
$$-\bar{P}(U,V)R(X,Y)W=0.$$

Putting $X=\xi$ and taking inner product on both sides of (3.4.6) by ξ, we get

(3.4.7) $$\eta\left(R(\xi,Y)\bar{P}(U,V)W\right)-\eta\left(\bar{P}\left(R(\xi,Y)U,V\right)W\right)-\eta\left(\bar{P}\left(U,R(\xi,Y)V\right)W\right)$$
$$-\eta\left(\bar{P}(U,V)R(\xi,Y)W\right)=0.$$

From this it follows that

(3.4.8)
$$'\bar{P}(U,V,W,Y)-\eta(Y)\eta\left(\bar{P}(U,V)W\right)-g(Y,U)\eta\left(\bar{P}(\xi,V)W\right)+\eta(U)\eta\left(\bar{P}(Y,V)W\right)$$
$$+\eta(W)\eta\left(\bar{P}(U,V)Y\right)-g(Y,V)\eta\left(\bar{P}(U,\xi)W\right)+\eta(V)\eta\left(\bar{P}(U,Y)W\right)=0,$$

where $'\bar{P}(U,V,W,Y)=g\left(\bar{P}(U,V)W,Y\right)$.

Putting $Y=U$ in (3.4.8), we get

(3.4.9) $$'\bar{P}(U,V,W,U)-g(U,U)\eta\left(\bar{P}(\xi,V)W\right)-g(U,V)\eta\left(\bar{P}(U,\xi)W\right)$$
$$+\eta(V)\eta\left(\bar{P}(U,U)W\right)+\eta(W)\eta\left(\bar{P}(U,V)U\right)=0.$$

Let $\{e_i : i=1,2,....,2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then the sum for $1\le i\le 2n+1$ of the relation (3.4.9) for $U=e_i$, yields

(3.4.10) $$\eta\left(\bar{P}(\xi,V)W\right)=\left(\frac{a+2nb}{2n}\right)S(V,W)-\frac{(a+2nb)r}{2n(2n+1)}g(V,W)$$
$$+\left[(a-b)\left\{\frac{r}{2n(2n+1)}-\left(\alpha^2-\beta^2\right)\right\}\right]\eta(V)\eta(W).$$

From (3.4.3) and (3.4.10), we get

(3.4.11)
$$S(V,W) = 2n(\alpha^2 - \beta^2) g(V,W) + \left[\frac{b}{a} \left\{ r - 2n(2n+1)(\alpha^2 - \beta^2) \right\} \right] \eta(V)\eta(Z).$$
This implies that the manifold is an η-Einstein manifold. Hence the theorem is proved. Again, taking $W = \xi$ in (3.4.11) and using (2.13), we get

(3.4.12) $$r = 2n(2n+1)(\alpha^2 - \beta^2).$$
Using (3.4.12) in (3.4.11), we obtain

(3.4.13) $$S(V,W) = 2n(\alpha^2 - \beta^2) g(V,W).$$
This leads to the following:

Theorem 3.5: A trans-Sasakian manifold satisfying the relation $R(X,Y).\bar{P} = 0$ is an Einstein manifold and also is a manifold of constant scalar curvature $2n(2n+1)(\alpha^2 - \beta^2)$.

Now, using (3.4.2), (3.4.3), (3.4.12) and (3.4.13) in (3.4.8), we obtain

$$'\bar{P}(U,V,W,Y) = g(\bar{P}(U,V)W,Y) = 0,$$
which yields

(3.4.14) $$\bar{P}(U,V)W = 0.$$
Therefore the trans-Sasakian manifold under consideration is pseudo-projectively flat. Hence we can state the next theorem:

Theorem 3.6: If in a trans-Sasakian manifold M of dimension $(2n+1), n > 0,$ the relation $R(X,Y).\bar{P} = 0$ holds, then the manifold is pseudo-projectively flat.

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