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POWERS X^N IN TERMS OF MODIFIED CHEBYSHEV POLYNOMIALS

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ABSTRACT
We exhibit two procedures to express x^n in terms of the shifted Chebyshev polynomials, which is useful to reduce the degree of a polynomial in the interval [0,1].

Keywords: Chebyshev-Lanczos polynomials

INTRODUCTION
In numerical analysis may be necessary to reduce, with small error, the degree of a polynomial in the interval [0, 1], which is possible employing the Modified Chebyshev polynomials $\overline{T}_r(x)$ defined by [1]:

$$\overline{T}_0(x) = \frac{1}{2}, \qquad \overline{T}_k(x) = T_k(2x-1), \qquad k = 1,2,... \tag{1}$$

where the first-kind Chebyshev polynomials $\overline{T}_r(x)$ are given by the recurrence relation [2-6]:

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \qquad k = 1,2,... \tag{2}$$

therefore

$$\begin{aligned} &\overline{T}_0(x) = \frac{1}{2}, \qquad \overline{T}_1(x) = 2x-1, \qquad \overline{T}_2(x) = 8x^2 - 8x + 1, \\ &\overline{T}_3(x) = 32x^3 - 48x^2 + 18x - 1, \qquad \overline{T}_4(x) = 128x^4 - 256x^3 - 32x + 1, \quad \text{etc.} \end{aligned} \tag{3}$$

In the mentioned reduction process we need the powers x^n in terms of $\overline{T}_r$, then from (3):

$$\begin{aligned} &x^0 = 2\overline{T}_0, \qquad x = \frac{1}{2}(2\overline{T}_0 + \overline{T}_1), \qquad x^2 = \frac{1}{8}(6\overline{T}_0 + 4\overline{T}_1 + \overline{T}_2), \\ &x^3 = \frac{1}{32}(20\overline{T}_0 + 15\overline{T}_1 + 6\overline{T}_2 + \overline{T}_3), \qquad x^4 = \frac{1}{128}(70\overline{T}_0 + 56\overline{T}_1 + 28\overline{T}_2 + 8\overline{T}_3 + \overline{T}_4), \quad \text{etc.} \end{aligned} \tag{4}$$

that is [1]:

$$\frac{1}{2}(4x)^n = \sum_{r=0}^{n} \binom{2n}{n-r} \overline{T}_r, \qquad n = 0,1,... \tag{5}$$

The next section exhibits an algorithm to obtain x^j in function of $\overline{T}_r$ if we know the corresponding expansion of x^{j-1}, and also another procedure which employs to (5) as a Newton's binomial expression.

x^n in terms of $\overline{T}_r$

We may write (5) in the form:

$$\begin{array}{c|cccccc} & \overline{T}_0 & \overline{T}_1 & \overline{T}_2 & \overline{T}_3 & \overline{T}_4 & \cdots \\ \frac{1}{2}(4x)^0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2}(4x)^1 & 2 & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2}(4x)^2 & 6 & 4 & 1 & 0 & 0 & \cdots \\ \frac{1}{2}(4x)^3 & 20 & 15 & 6 & 1 & 0 & \cdots \\ \frac{1}{2}(4x)^4 & 70 & 56 & 28 & 8 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{6}$$

or in function of the columns vectors $(\frac{1}{2}(4x)^j)$ and $(\overline{T}_r)$ for a given n:

$$\begin{pmatrix} \frac{1}{2}(4x)^0 \\ \frac{1}{2}(4x)^1 \\ \vdots \\ \frac{1}{2}(4x)^n \end{pmatrix} = \underset{\sim}{A} \cdot \begin{pmatrix} \overline{T}_0 \\ \overline{T}_1 \\ \vdots \\ \overline{T}_n \end{pmatrix} \tag{7}$$

where $\underset{\sim}{A}$ is the $(n+1) \times (n+1)$ triangular matrix of coefficients appearing in (6):

$$\underset{\sim}{A} = (a_{jr}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 4 & 1 & 0 & 0 & \cdots \\ 20 & 15 & 6 & 1 & 0 & \cdots \\ 70 & 56 & 28 & 8 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad j, r = 0,1,...n \tag{8}$$

then $(\overline{T}_r) = \underset{\sim}{A}^{-1} \cdot (\frac{1}{2}(4x)^r)$ reproduces (3).

The relations (5) and (7) imply that:

$$a_{jr} = \binom{2n}{n-r}, \qquad j, r = 0,1,... \tag{9}$$

thus

$$a_{jj} = 1, \qquad a_{jr} = 0, \qquad r > j \tag{10}$$

and we can prove the following properties not found explicitly in the literature:

$$\begin{aligned} a_{j+1,0} &= 2(a_{j0} + a_{j1}), \qquad j = 0,1,2,... \\ a_{jr} &= a_{j-1,r-1} + 2a_{j-1,r} + a_{j-1,r+1}, \qquad r, j = 1,2,3,... \end{aligned} \tag{11}$$

The formulae (11) permit to construct the row *j* of $\underset{\sim}{A}$ if we know its row *j-1*, and they represent an algorithm to express x^n in terms of $(\overline{T}_r)$ whose systematic use minimize the amount of arithmetical computations involved in (5).

On the other hand, the expansion (5) can be written as:

$$\frac{1}{2}(4x)^n = \sum_{k=0}^{n} \binom{2n}{k} \overline{T}_{n-k} = \sum_{k=0}^{2n} \binom{2n}{k} \overline{T}^{n-k} \tag{12}$$

where we use the notation:

$$\overline{T}^{-j} = 0, \qquad j = 1,2,..., \qquad \overline{T}^r \equiv \overline{T}_r, \qquad r = 0,1,2,... \tag{13}$$

very employed in Gregory-Newton and Stirling interpolations [7].

Thus (12) adopts the form of a Newton's binomial expression:

$$\frac{1}{2}(4x)^n = \frac{1}{\overline{T}^n}\sum_{k=0}^{2n}\binom{2n}{k}\overline{T}^{2n-k} = \frac{1}{\overline{T}^n}(1+\overline{T})^{2n} \tag{14}$$

which is a procedure alternative to (11) to obtain x^n in function of $\overline{T}_r$. For example:

$$\frac{1}{2}(4x)^2 = \frac{1}{\overline{T}^2}(1+\overline{T})^4 = \frac{1}{\overline{T}^2}(1+4\overline{T}+6\overline{T}^2+4\overline{T}^3+\overline{T}^4),$$
$$= \overline{T}^{-2}+4\overline{T}^{-1}+6\overline{T}^0+4\overline{T}+\overline{T}^2 = 6\overline{T}_0+4\overline{T}_1+\overline{T}_2, \quad \text{etc.}$$

in according with (6). The relation (14) may be easily manipulated by a computer via some symbolic language as MAPLE.

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