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FIXED POINT THEOREM FOR A PAIR OF SELF MAPS SATISFYING A GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE

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ABSTRACT
In this paper, we establish a fixed point theorem for a pair of self maps satisfying a general contractive condition of integral type. This theorem extends and generalizes some early results of Boikanyo[2].

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Key Words: Lebesgue-integrable map, Complete metric space, Common fixed point.

1. INTRODUCTION
The first well known result on fixed points for contractive map was Banach – Caccioppoli theorem, published in 1922 (see [1], [4]). In general setting of complete metric space, Smart ([11]) presented the following result.

Theorem 1.1. Let (X, d) be a complete metric space, $c \in [0, 1)$, and let $T : X \to X$ be a map such that for each $x, y \in X$,

$$d(Tx, Ty) \le c\, d(x, y)$$

Then, T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

After this classical result, many theorems dealing with maps satisfying various types of contractive inequalities have been established (see for details [2], [5]-[10], [12]).
In 2002, Branciari ([3]) obtained the following theorem.

Theorem 1.2. Let (X, d) be a complete metric space, $c \in [0,1)$, and let $T : X \to X$ be a map such that for each $x, y \in X$,

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le c \int_0^{d(x,y)} \varphi(t)\,dt$$

where $\varphi : R^+ \to R^+$ is a Lebesgue - integrable map which is summable, non negative and such that $\int_0^{\varepsilon} \varphi(t)\, dt > 0$ for each $\varepsilon > 0$. Then, T has a unique fixed point $z \in X$ and for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

In 2007, Boikanyo [2] proved some fixed point theorems for a self map satisfying a general contractive condition of integral type as an extension of Branciari's theorem. In [3], it was mentioned that one can generalize other results related to contractive conditions of some kind, such as in [8].

The main purpose of our paper is to obtain some results for a pair of self maps satisfying a general contractive condition of integral type.

Throughout this paper, N denotes the set of natural numbers.

2. Main Results

Theorem 2.1. Let (X, d) be a complete metric space. Let $a_i (i = 1,2,3,4,5)$ be non negative real numbers satisfying $\sum_{i=1}^{5} a_i < 1$, T_1 and T_2 be a pair of self maps of the metric space X such that for each $x, y \in X$

$$\int_0^{d(T_1x,T_2y)} \varphi(t)\,dt \le a_1 \int_0^{d(x,y)} \varphi(t)\,dt + a_2 \int_0^{d(x,T_1x)} \varphi(t)\,dt + a_3 \int_0^{d(y,T_2y)} \varphi(t)\,dt + a_4 \int_0^{d(x,T_2y)} \varphi(t)\,dt + a_5 \int_0^{d(y,T_1x)} \varphi(t)\,dt \tag{2.1}$$

where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable map which is summable, non-negative and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$. Then T_1 and T_2 have a unique common fixed point $z \in X$.

Proof. Let x_0 be any point of X.

Define $x_{2n-1} = T_1 x_{2n-2}$

$x_{2n} = T_2 x_{2n-1}$ where $n \in N$

We claim that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

(2.2)

To prove (2.2), we require to show that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt \le r^n \int_0^{d(x_0,x_1)} \varphi(t)\,dt \quad \text{where } r = \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - a_2 - a_3 - a_4 - a_5} < 1$$

For this, by interchanging x with y and T_1 with T_2 in (2.1), we obtain

$$\int_0^{d(T_2y,T_1x)}\varphi\ (t)\,dt \le a_1\int_0^{d(y,x)}\varphi(t)\,dt + a_2\int_0^{d(y,T_2y)}\varphi(t)\,dt$$
$$+ a_3\int_0^{d(x,T_1x)}\varphi(t)\,dt + a_4\int_0^{d(y,T_1x)}\varphi(t)\,dt + a_5\int_0^{d(x,T_2y)}\varphi(t)\,dt \qquad (2.3)$$

Now, from (2.1), (2.3) and using symmetric property, we obtain

$$\int_0^{d(T_1x,T_2y)}\varphi(t)\,dt \le a_1\int_0^{d(x,y)}\varphi(t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(x,T_1x)}\varphi(t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(y,T_2y)}\varphi(t)\,dt$$

$$+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x,T_2y)}\varphi(t)\,dt + \left(\frac{a_4+a_5}{2}\right)\int_0^{d(y,T_1x)}\varphi\ (t)\,dt \qquad (2.4)$$

Using (2.4) for odd n, we obtain

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt = \int_0^{d(T_1x_{n-1},T_2x_n)}\varphi(t)\,dt$$
$$\le a_1\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_{n-1},T_1x_{n-1})}\varphi(t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_n,T_2x_n)}\varphi(t)\,dt$$
$$+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_{n-1},T_2x_n)}\varphi(t)\,dt + \left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_n,T_1x_{n-1})}\varphi(t)\,dt$$
$$= a_1\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt$$
$$+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_{n-1},x_{n+1})}\varphi(t)\,dt + \left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_n,x_n)}\varphi(t)\,dt$$

Again, using (2.4) for even n, we obtain

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt = \int_0^{d(T_2x_{n-1},T_1x_n)}\varphi(t)\,dt$$
$$\le a_1\int_0^{d(x_{n-1},x_n)}\varphi\ (t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_{n-1},T_2x_{n-1})}\varphi(t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_n,T_1x_n)}\varphi(t)\,dt$$
$$+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_{n-1},T_1x_n)}\varphi(t)\,dt + \left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_n,T_2x_{n-1})}\varphi(t)\,dt$$
$$= a_1\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt + \left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt$$

$$+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_{n-1},x_{n+1})}\varphi(t)\,dt+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_n,x_n)}\varphi(t)\,dt$$

From the above two cases, one can see that

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt \le a_1\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+\left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+\left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt$$

$$+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_{n-1},x_{n+1})}\varphi(t)\,dt+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_n,x_n)}\varphi(t)\,dt$$

$$\le a_1\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+\left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+\left(\frac{a_2+a_3}{2}\right)\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt$$

$$+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+\left(\frac{a_4+a_5}{2}\right)\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt$$

It follows that $\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt \le \left(\frac{2a_1+a_2+a_3+a_4+a_5}{2-a_2-a_3-a_4-a_5}\right)\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt$

$$= r\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt$$

$\le r^n\int_0^{d(x_0,x_1)}\varphi(t)\,dt \to 0$ as $n\to\infty$ since $r<1$, owing to the assumption $\sum_{i=1}^{5} a_i < 1$. Therefore, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in *X*. Let $m > n$ where $m,n\in N$

Without any loss of generality, two cases arise:

(i) *m* is even when *n* is odd

and (ii) *m* is odd when *n* is even.

Case I: We choose *n* and *m* to be odd and even respectively

Then we have

$$\int_0^{d(x_n,x_m)}\varphi(t)\,dt = \int_0^{d(T_1x_{n-1},T_2x_{m-1})}\varphi(t)\,dt$$

$$\le a_1\int_0^{d(x_{n-1},x_{m-1})}\varphi(t)\,dt + a_2\int_0^{d(x_{n-1},T_1x_{n-1})}\varphi(t)\,dt + a_3\int_0^{d(x_{m-1},T_2x_{m-1})}\varphi(t)\,dt$$

$$+a_4\int_0^{d(x_{n-1},T_2x_{m-1})}\varphi(t)\,dt + a_5\int_0^{d(x_{m-1},T_1x_{n-1})}\varphi(t)\,dt$$

$$= a_1\int_0^{d(x_{n-1},x_{m-1})}\varphi(t)\,dt + a_2\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt + a_3\int_0^{d(x_{m-1},x_m)}\varphi(t)\,dt$$

$$+a_4\int_0^{d(x_{n-1},x_m)}\varphi(t)\,dt+a_5\int_0^{d(x_{m-1},x_n)}\varphi(t)\,dt$$

Case II: We choose *n* and *m* to be even and odd respectively.

Then we have

$$\int_0^{d(x_n,x_m)}\varphi(t)\,dt=\int_0^{d(T_2x_{n-1},T_1x_{m-1})}\varphi(t)\,dt$$

$$\le a_1\int_0^{d(x_{n-1},x_{m-1})}\varphi(t)\,dt+a_2\int_0^{d(x_{n-1},T_2x_{n-1})}\varphi(t)\,dt+a_3\int_0^{d(x_{m-1},T_1x_{m-1})}\varphi(t)\,dt$$

$$+a_4\int_0^{d(x_{n-1},T_1x_{m-1})}\varphi(t)\,dt+a_5\int_0^{d(x_{m-1},T_2x_{n-1})}\varphi(t)\,dt$$

$$=a_1\int_0^{d(x_{n-1},x_{m-1})}\varphi(t)\,dt+a_2\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+a_3\int_0^{d(x_{m-1},x_m)}\varphi(t)\,dt$$

$$+a_4\int_0^{d(x_{n-1},x_m)}\varphi(t)\,dt+a_5\int_0^{d(x_{m-1},x_n)}\varphi(t)\,dt$$

From both the cases, we have

$$\int_0^{d(x_n,x_m)}\varphi(t)\,dt\le a_1\int_0^{d(x_{n-1},x_{m-1})}\varphi(t)\,dt+a_2\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+a_3\int_0^{d(x_{m-1},x_m)}\varphi(t)\,dt$$

$$+a_4\int_0^{d(x_{n-1},x_m)}\varphi(t)\,dt+a_5\int_0^{d(x_{m-1},x_n)}\varphi(t)\,dt$$

$$\le a_1\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+a_1\int_0^{d(x_n,x_m)}\varphi(t)\,dt+a_1\int_0^{d(x_m,x_{m-1})}\varphi(t)\,dt$$

$$+a_2\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+a_3\int_0^{d(x_{m-1},x_m)}\varphi(t)\,dt+a_4\int_0^{d(x_n,x_m)}\varphi(t)\,dt$$

$$+a_4\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+a_5\int_0^{d(x_{m-1},x_m)}\varphi(t)\,dt+a_5\int_0^{d(x_m,x_n)}\varphi(t)\,dt$$

Therefore

$$\int_0^{d(x_n,x_m)}\varphi(t)\,dt\le\left(\frac{a_1+a_2+a_4}{1-a_1-a_4-a_5}\right)\int_0^{d(x_{n-1},x_n)}\varphi(t)\,dt+\left(\frac{a_1+a_3+a_5}{1-a_1-a_4-a_5}\right)\int_0^{d(x_{m-1},x_m)}\varphi(t)\,dt$$

$$\le\left(\frac{a_1+a_2+a_4}{1-a_1-a_4-a_5}\right)r^{n-1}\int_0^{d(x_0,x_1)}\varphi(t)\,dt+\left(\frac{a_1+a_3+a_5}{1-a_1-a_4-a_5}\right)r^{m-1}\int_0^{d(x_0,x_1)}\varphi(t)\,dt$$

$$\to 0\ \text{as}\ n,m\to\infty,\ \text{since}\ r<1$$

Hence, $\{x_n\}$ is a Cauchy sequence in the complete metric space *X*, so it is convergent in *X*.

Let its limit be z, i.e. $\lim_{n\to\infty} x_n = z$. We show that $T_1 z = T_2 z = z$.

Now, we have

$$\int_0^{d(x_{2n},T_1z)} \varphi(t)\,dt = \int_0^{d(T_2x_{2n-1},T_1z)} \varphi(t)\,dt$$

$$\le a_1\int_0^{d(x_{2n-1},z)} \varphi(t)\,dt + a_2\int_0^{d(x_{2n-1},T_2x_{2n-1})} \varphi(t)\,dt + a_3\int_0^{d(z,T_1z)} \varphi(t)\,dt$$

$$+a_4\int_0^{d(x_{2n-1},T_1z)} \varphi(t)\,dt + a_5\int_0^{d(z,T_2x_{2n-1})} \varphi(t)\,dt$$

$$= a_1\int_0^{d(x_{2n-1},z)} \varphi(t)\,dt + a_2\int_0^{d(x_{2n-1},x_{2n})} \varphi(t)\,dt + a_3\int_0^{d(z,T_1z)} \varphi(t)\,dt$$

$$+a_4\int_0^{d(x_{2n-1},T_1z)} \varphi(t)\,dt + a_5\int_0^{d(z,x_{2n})} \varphi(t)\,dt$$

Taking the limit as $n \to \infty$, we get

$$\int_0^{d(z,T_1z)} \varphi(t)\,dt \le a_3\int_0^{d(z,T_1z)} \varphi(t)\,dt + a_4\int_0^{d(z,T_1z)} \varphi(t)\,dt$$

$$\Rightarrow \int_0^{d(z,T_1z)} \varphi(t)\,dt = 0$$

$$\Rightarrow z = T_1 z$$

Similarly, it may be shown that $T_2 z = z$. Thus T_1 and T_2 have a common fixed point.

For uniqueness, if possible, let w be another common fixed point of T_1 and T_2 such that $w \ne z$.

Now, we have

$$\int_0^{d(z,w)} \varphi(t)\,dt = \int_0^{d(T_1z,T_2w)} \varphi(t)\,dt$$

$$\le a_1\int_0^{d(z,w)} \varphi(t)\,dt + a_2\int_0^{d(z,T_1z)} \varphi(t)\,dt + a_3\int_0^{d(w,T_2w)} \varphi(t)\,dt$$

$$+a_4\int_0^{d(z,T_2w)} \varphi(t)\,dt + a_5\int_0^{d(w,T_1z)} \varphi(t)\,dt$$

$$= a_1\int_0^{d(z,w)} \varphi(t)\,dt + a_4\int_0^{d(z,w)} \varphi(t)\,dt + a_5\int_0^{d(w,z)} \varphi(t)\,dt$$

$\Rightarrow \int_0^{d(z,w)} \varphi(t)\,dt = 0$, a contradiction. Hence, $z = w$.

Thus T_1 and T_2 have a unique common fixed point. This completes the proof. □

Corollary 2.2. Let (X, d) be a complete metric space. Let a , b , c be non negative real numbers satisfying $a + b + c < 1$, T_1 and T_2 be a pair of self maps of the metric space X into itself such that for each $x, y \in X$,

$$\int_0^{d(T_1x,T_2y)} \varphi(t)dt \le a\int_0^{d(x,T_1x)} \varphi(t)dt + b\int_0^{d(y,T_2y)} \varphi(t)dt + c\int_0^{d(x,y)} \varphi(t)dt$$

(2.5)

where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable map which is summable, non-negative and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$. Then T_1 and T_2 have a unique common fixed point $z \in X$.

Proof. Since the contractive condition (2.5) is obviously a special case of (2.1) by setting $a_1 = c,\ a_2 = a,\ a_3 = b$ and $a_4 = a_5 = 0$, the result follows immediately from Theorem 2.1.□

Corollary 2.3. Let (X , d) be a complete metric space. Let a, b, c be non negative real numbers satisfying $a + b + c < 1$, T_1 and T_2 be a pair of self maps of the metric space X such that for each $x, y \in X$,

$$\int_0^{d(T_1x,T_2y)} \varphi(t)dt \le a\int_0^{d(x,T_2y)} \varphi(t)dt + b\int_0^{d(y,T_1x)} \varphi(t)dt + c\int_0^{d(x,y)} \varphi(t)dt$$

(2.6)

where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable map which is summable, non-negative and such that $\int_0^{\varepsilon} \varphi(t)dt > 0$ for each $\varepsilon > 0$. Then T_1 and T_2 have a unique common fixed point $z \in Z$.

Proof. Since the contractive condition (2.6) is also a special case of (2.1) by letting $a_1 = c$, $a_4 = a,\ a_5 = b$ and $a_2 = a_3 = 0$,the result follows immediately from Theorem 2.1.□

Remark 2.4. We give some remarks which clarify the connection between our results and the results obtained in [2].

(i) Theorem 1 and 2(cf. [2]) are special cases of Corollary 2.2 and 2.3 respectively with $T_1 = T_2$, $a = b$ and $c = 0$.

(ii) By taking $T_1 = T_2$, Corollary 2.2 and 2.3 reduce Theorem 3 and 4 (cf. [2]) respectively.

(iii) Theorem 5 (cf. [2]) is a consequence of Theorem 2.1 if we take $T_1 = T_2$.

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