KATHMANDU UNIVERSITY JOURNAL OF SCIENCE, ENGINEERING AND TECHNOLOGY
VOL. 5, NO. 1. JANUARY, 2009, PP. 23- 30

PARSEVAL'S IDENTITY FOR LOW-DIMENSIONAL NILPOTENT LIEGROUPS $G_{5,1}$, $G_{5,2}$, $G_{5,3}$, AND $G_{5,5}$

C.R. Bhatta
Central Department of Mathematics
Tribhuvan University, Kirtipur, Kathmandu, NEPAL

Corresponding author: crbhatta@yahoo.com

Received 16 December, 2008: Revised 11 January, 2009

ABSTRACT

We prove the Parseval's identity for low-dimensional nilpotent Lie groups such as $G_{5,1}$, $G_{5,2}$, $G_{5,3}$ and $G_{5,5}$ respectively which are important for proving Hardy uncertainty principles type result.

Keywords: Fourier transform, Hilbert schmidt norm, Kernel function.

INTRODUCTION

Let g be an n-dimensional real nilpotent Lie algebra and let G = exp g be the associated, connected and simply connected nilpotent Lie group. Let $\{x_1, ..., x_n\}$ be a strong Malcev basis of g through the ascending central series of g. In particular, RX_1 is contained in the centre of g. We introduce a norm function on G by setting for

$x = \exp(x_1X_1 + \ldots + x_nX_n) \in G,\ x_j \in R$

$$\|x\| = \left(x_1^2 + \Lambda + x_n^2\right)^{1/2}$$

The composed map

$$R^n \to g \to G,\ (x_1, \ldots, x_n) \to \sum_{j=1}^{n} x_j X_i \to \exp\left(\sum_{j=1}^{n} x_j X_j\right)$$

is a diffeomorphism and maps Lebesgue measure on R^n to Haar measure on G. In this manner we shall always identify g and sometimes G, as sets with R^n. Thus measurable (integrable) functions on G can be viewed as such functions on R^n.

Let g^* denote the vector space dual of g and $\{X_1^*, \Lambda\ X_n^*\}$ the basis of g^* which is dual to $\{X_1, \ldots, X_n)$. Then $\{X_1^*, \Lambda\ X_n^*\}$ is Jordan-Holder basis for the Coadjoint action of G on g^*. We shall identify g^* with R^n via the map $\xi = (\xi_1, \ldots \xi_n) \to \sum_{j=1}^{n} \xi_j X_j^*$ and on g^* we introduce the Euclidean norm relative to the basis $\{X_1^*, \Lambda\ X_n^*\}$, that is, $\left\|\sum_{j=1}^{n} \xi_j X_j^*\right\| = \left(\xi_1^2 + \Lambda\ \xi_n^2\right)^{1/2} = \|\xi\|$

For an operator T in a Hilbert space such that T^*T is a trace class $\|T\|_{HS}$ will denote the Hilbert schmidt norm of T.

THREAD LIKE NILPOTENT LIE GROUPS

For $n \geq 3$, leg g_n be the n-dimenstional real nilpotent Lie algebra with basis $X_1, \ldots, X_n$ and non trivial Lie brackets $[X_1, x_{n-1}] = X_{n-2}, \ldots, [X_n, X_2] = X_1$

g_n is a (n - 1) step nilpotent and is a semidirect product of RX_n and the abelian ideal $\sum_{j=1}^{n-1} RX_j$. Note that g_3 is the Heisenberg Lie algebra. Let $G_n = \exp g_n$.

For $\xi = \sum_{j=1}^{n-1} \xi_j X_j^* \in g_n^*$, the coadjoint action of G_n is given by

$$Ad^* (\exp (tX_n)) \xi = \sum_{j=1}^{n-1} P_j (\xi, t) X_j^*,$$

where, for $i \leq j \leq n - 1$, $P_j (\xi, t)$ is the polynomial in t defined by

$$P_j(\xi, t) = \sum_{k=1}^{j-1} \left(\tfrac{1}{k!}\right) (-1)^k t^k \xi_{j-k}.$$

The orbit of ξ is generic with respect to the basis $\{X_1^*, \Lambda, X_n^*\}$ if and only if $\xi_1 \neq 0$, and the jumping indices are 2 to n. The cross section X_{ξ_1} for the set of generic orbits is given by

$$X_{\xi_1} = \{\xi = (\xi_1, 0, \xi_3, \ldots, \xi_{n-1}, 0): \xi_1 \in R, \xi_1 \neq 0\}$$

For $\xi \in g_n^*$, let π_ξ denote the ireducible representation of G_n associated with ξ. Then the mapping $\xi \to \pi_\xi$ is bijection of X_{ξ_1} and the set of all generic irreducible representations. Plancherel measure on $\hat{G}_n$ is supported by these π_ξ. Denoting by F the Fourier transform on R^{n-1}, it follows that the Hilbert schmidt norm of the operator $\pi_\xi (f)$, $f \in L^1 \cap L^2 (G_n)$ is given by,

$$\|\pi_\xi (f)\|_{HS}^2 = \int_{R^2} | Ff(p_1(\xi, t), \ldots, p_{n-1} (\xi, t), t - s |^2 \, dsdt$$

The following group of lower dimensions such as $G_{5,1}$, $G_{5,2}$, $G_{5,3}$ and $G_{5,5}$ etc are found in [8].

PARSEVAL IDENTITY FOR $G_{5,1}$

Let $G = G_{5,1} = R^5$

$$(x_1, \ldots, x_5)(y_1, \ldots, y_5) = (x_1 + y_1 + x_3y_2 + x_5y_4, x_2 + y_2, x_3 + y_3, x_4 + y_4, x_5 + y_5)$$

and $(x_1, \ldots, x_5)^{-1} = (-x_1 + x_2x_3 + x_4x_5, -x_2, -x_3, -x_4, -x_5)$

For $y \in R^5$

$$\pi_{\xi_1} (f) \phi (y) = \int_{R^5} f(x_1,x_2,x_3,x_4,x_5) \, \pi_{\xi_1} (-x_1 + x_2x_3 + x_4x_5, -x_2, -x_3, -x_4, -x_5) \phi (y_1, y_2)$$

$$= \int_{R^5} f(x) \exp [2\pi i (-x_1 + x_2x_3 + x_4x_5 + x_2x_1 + x_4y_2) \xi_1] \phi (y_1 + x_3, y_2 + x_5) \, dx$$

$$x_3 \to x_3 - y_1, \; x_5 \to x_5 - y_2$$

$$= \int_{R^5} f(x_1, x_2, x_3 - y_1, x_4, x_5 - y_2) \exp [2\pi i (-x_1 + x_2 (x_3 - y_1)$$

$$+ x_4((x_5 - y_2) + x_2y_1 + x_4y_2) \xi_1] \phi (x_3, x_5)dx$$

$$= \int_{R^5} f(x_1, x_2, x_3 - y_1, x_4, x_5 - y_2) \exp [2\pi i (-x_1 + x_2x_3 - x_2y_1 + x_4y_5 - x_4y_3 + x_2y_1 + x_4y_2) \xi_1]$$

$$\phi (x_3, x_5) dx$$

$$= \int_{R^5} f(x_1,x_2,x_3 - y_1,x_4, x_5 - y_2) \exp [-2\pi i (x_1 - x_2x_3 - x_4y_5)\xi_1] \phi (x_3,x_5)dx$$

$$K_{\xi_1}^f (y_1, y_2, x_3, x_5) = \int_{R^5} f(x_1, x_2, x_3 - y_1, x_4, x_5 - y_2)$$

$$\exp [-2\pi i (x_1\xi_1 - x_2x_3\xi_1 - x_4x_5\xi_1)] dx_1dx_2dx_4$$

$$= F_{124}\, f(\xi_1, -x_3\xi_1, x_3 - y_1, - x_5\xi_1, x_5 - y_2)$$

$$\|\pi_{\xi_1}(f)\|_{HS}^2 = \int_{R^5} | K_{\xi_1}^f (y_1, y_2, x_3, x_5)|^2 dy_1dy_2dx_3dx_5$$

$$= \int_{R^4} |F_{124}\, f(\xi_1, - x_3\xi_1, x_3 - y_1, -x_5\xi_1, x_5 - y_2|^2 dy_1dy_2dx_3dx_5$$

$$x_3 \to \frac{-1}{\xi_1} x_3, x_5 \to \frac{-1}{\xi_1} x_5$$

$$= \frac{1}{\xi_1^2} \int_{R^4} |F_{124}\, f(\xi_1, x_3, y_1, x_5, y_2)|^2 dy_1dy_2dx_3dx_5$$

$$= \frac{1}{\xi_1^2} \int_{R^4} |F_{14} f(\xi_1, u, y_1, x_5, y_2|^2 dy_1dy_2dxdx_5$$

$$= \frac{1}{\xi_1^2} \int_{R^4} |F_1\, f(\xi_1, u, y_1, w, y_2|^2 dy_1dy_2dudw$$

PARSEVAL'S IDENTITY FOR $G_{5,2}$

Let $G = G_{5,2} = R^5$

$(x_1, \ldots, x_5) (y_1, \ldots, y_5) = (x_1 + y_1 + x_5y_3, x_2 + y_2 + x_5y_4, x_3 + y_3, x_4 + y_4, x_5 + y_5)$

and $(x_1, \ldots, x_5)^{-1} = (-x_1 + x_3x_5, -x_2 + x_4x_5, -x_3, -x_4, -x_5)$

For $y \in R^5$

$$\pi_{\xi_1,\xi_2,\xi_4} (f)\phi (y) = \int_{R^5} f(x_1,x_2,x_3,x_4,x_5)\ \pi_{\xi_1,\xi_2,\xi_4}\ (-x_1 + x_3x_5, -x_2 + x_4x_5, -x_3, -x_4, -x_5) \phi (y)dx$$

$$= \int_{R^5} f(x_1,x_2,x_3,x_4,x_5) \exp [2\pi i (-x_1 + x_3x_5 + x_3y)\xi_1 + (-x_2 + x_4x_5 + x_4y)\xi_2 - x_4\xi_4] \phi (y + x_5)dx$$

$$x_5 \to x_5 - y$$

$$= \int_{R^5} f(x_1,x_2,x_3,x_4,x_5 - y) \exp [2\pi i (-x_1 + x_3 (x_5 - y) + x_3y)\xi_1$$

$$+ (-x_2 + x_4(x_5 - y) + x_4y)\xi_2 - x_4\xi_4] \phi (x_5)dx$$

$$= \int_{R^5} f(x_1,x_2,x_3,x_4,x_5 - y)\exp [2\pi i[(-x_1 - x_3y + x_3x_5 + x_3y)\xi_1$$

$$+ (-x_2 -x_4y + x_4x_5 + x_4y)\xi_2 - x_4\xi_4] \phi (x_5)dx$$

$$K^f_{\xi_1,\xi_2,\xi_4}(y, x_5) = \int_{R^4} f(x_1, x_2, x_3, x_4, x_5 - y) \exp [-2\pi i(x_3\xi_1 - x_3x_5\xi_1 + x_2\xi_2 - x_4x_5\xi_4+x_4\xi_3)dx_1dx_2dx_3dx_4$$

$$= F_{1234} (\xi_1, \xi_2, - x_5\xi_1, -x_5\xi_2 + \xi_4, x_5 - y)$$

$$\|\pi_{\xi_1,\xi_2,\xi_4}(f)\|^2_{HS} = \int_{R^2}\left|K^f_{\xi_1,\xi_2,\xi_4}(y,x_5)\right|^2 dydx_5$$

$$= \int_{R^2} |F_{1234}f(\xi_1, \xi_2, -x_5\xi_1, -x_5\xi_2 + \xi_4, x_5 - y)|^2 dydx_5$$

$$x_5 \to \frac{-1}{\xi_1}x_5$$

$$= \frac{1}{|\xi_1|^2}\int_{R^2}\left|F_{1234}f\left(\xi_1,\xi_2,x_5,\frac{x_5\xi_2}{\xi_1}+\xi_4,\frac{-1}{\xi_1}x_5 - y\right)\right|^2 dydx_5$$

$$y \to -y - \frac{1}{\xi_1}x_5$$

$$= \frac{1}{|\xi_1|^2}\int_{R^2}\left|F_{1234}f\left(\xi_1,\xi_2,x_5\frac{x_5\xi_2}{\xi_1}+\xi_4,y\right)\right|^2 dydx_5$$

$$= \frac{1}{|\xi_1|^2}\int_{R^2}\left|F_{124}f\left(\xi_1,\xi_2,w,\frac{w\xi_2}{\xi_1}+\xi_4,u\right)\right|^2 dudw$$

PARSEVAL'S IDENTITY FOR $G_{5,3}$

Let $G = G_{5,3} = R^5$

$$(x_1, \ldots, x_5) = (y_1, \ldots, y_5) = (x_1 + y_1 + x_4y_3 + x_5y_2 + \frac{1}{2}x_5^2y_4, x_2 + y_2 + x_5y_4, x_3 + y_3,x_4 + y_4,x_5 + y_5)$$

$$(x_1, \ldots, x_5)^{-1} = (-x_1 + x_2x_5 + x_3x_4 - \frac{1}{2}x_4x_5^2, - x_2 + x_4x_5, - x_3, - x_4, - x_5)$$

For $y \in R^2$

$$\pi_{\xi_1}(f)\phi(y) = \int_{R^5} f(x)\ \pi_{\xi_1}\left(-x_1 + x_2x_5 + x_3x_4 - \frac{1}{2}x_4x_5^2 - x_2 + x_4x_5, -x_3, -x_4, -x_5\right)\phi(y_1, y_2)dx$$

$$= \int_{R^5} f(x)\exp\Big[2\pi i\Big(-x_1 + x_2x_5 + x_3x_4, -\frac{1}{2}x_4x_5^2 - x_3x_4 - x_4y_1 - (-x_2 + x_4x_5)y_2$$

$$-\frac{1}{2}x_4y_2^2\Big)\Big]\phi(y_1 + x_3, y_2 + x_5)$$

$$x_3 \to x_3 - y_1, x_5 \to x_5 - y_2$$

$$= \int_{R^5} f(x_1,x_2,x_3, -y_1,x_4,x_5 - y_2)\exp\ [2\pi i(-x_1 + x_2x_5 + (x_3 - y_1)x_4 - \frac{1}{2}x_4$$

$$(x_5 - y_2)^2 - (x_3 - y_1)x_4 - x_4y_1 - (-x_2 + x_4\,(x_5 - y_2))y_2 - \frac{1}{2}x_4\,y_2^2\,]\phi\,(x_3, x_5)dx$$

$$= \int_{R^5} f(x_1,x_2,x_3 - y_1,x_4,x_5 - y_2)\exp\ [2\pi i(-x_1 + x_2x_5 + x_3x_4 - y_1x_4$$

$$-\frac{1}{2}x_4(x_5 - y_2)^2 - x_3x_4 + x_4y_1 - x_4y_1 + x_2y_2 - x_4y_2(x_5 - y_2) - \frac{1}{2}x_4\,y_2^2\,]\xi_1\phi\,(x_3, x_5)dx$$

$$= \int_{R^5} f(x_1,x_2,x_3 - y_1,x_4,x_5 - y_2)\exp\ [2\pi i\ (-x_1 + x_2\,(x_5 + y_2) - y_1x_4 - \frac{1}{2}x_4\,x_5^2\,)\xi_1]\ \phi\,(x_3, x_5)dx$$

$$K_{\xi_1}^{f}(y_1,y_2,x_3,x_5) = \int_{R^5} f(x_1,x_2,x_3 - y_1x_4,x_5 - y_2)\exp\ [(-2\pi i\ (x_1\xi_1 - x_2\xi_1\ (x_5+y_2) + x_4(y_1+ \frac{1}{2}x_5^2)\xi_1]$$
$$dx_1dx_2dx_4$$

$$= F_{124}f\left(\xi_1, -\xi_1(x_5 + y_2), x_3 - y_1, \left(y_1 + \frac{1}{2}x_5^2\right)\xi_1, x_5 - y_2\right)$$

$$\left\|\pi_{\xi_1}(f)\right\|_{HS}^2 = \int_{R^4}\left|K_{\xi_1}^{f}(y_1, y_1, x_3, x_5\right|^2 dy_1dy_2dx_3dx_5$$

$$\int_{R^4}\left|F_{124}\left(\xi_1, -\xi_1(x_5 + y_1), x_3 - y_1, \left(y_1 + \frac{1}{2}x_5^2\right)\xi_1, x_5 - y_2\right)\right|^2 dy_1dy_2dx_3dx_5$$

$$x_3 \to x_3 + y_1,\ x_5 \to x_5 + y_2$$

$$= \int_{R^4} |F_{124}f|\left(\xi_1, -\xi_1(x_5 + 2y_2), x_3, (y_1 + \frac{1}{2}(x_5 + y_2)^2)\xi_1, x_5\right)|^2\,dy_1dy_2dx_3dx_5$$

$$y_1 \to y_1 - \frac{1}{2}(x_5 + y_2)^2$$

$$= \int_{R^4} |F_{124}f(\xi_1, -\xi_1(x_5 + 2y_2), x_3, y_1\xi_1, x_5)|^2\ dy_1dy_2dx_3dx_5$$

$$y_2 \to \frac{1}{\xi_1}\left(\frac{1}{2}y_2 - x_5\right),\ y_1 \to \frac{1}{\xi_1}y_1$$

$$= \frac{1}{2\xi_1^2}\int_{R^4} |F_{124}f(\xi_1,y_2,x_3y_1,x_5)|^2 dy_1dy_2dx_3dx_5$$

$$= \frac{1}{2\xi_1^2}\int_{R^4} |f_1(\xi_1,u,x_3,\upsilon,x_5)|^2 dyd\upsilon dx_3dx_5.$$

PARSEVAL'S IDENTITY FOR $G_{5,5}$

Let $G = G_{5,5} = R^5$

$$(x_1, \ldots, x_5)\,(y_1, \ldots, y_5) = \left(x_1 + y_1 + x_5y_2 + \frac{1}{2}x_5^2y_3 + \frac{1}{6}x_5^3y_4,\ x_2 + y_2 \right.$$

$$\left. + x_5y_3 + \frac{1}{2}x_5^2y_4, x_3 + y_3 + x_5y_4, x_4 + y_4, x_5 + y_5\right)$$

$$(x_1, \ldots, x_5)^{-1} = \left(-x_1 + x_2x_5 - \frac{1}{2}x_3x_5^2 + \frac{1}{6}x_4x_5^3, -x_2 + x_3x_5 - \frac{1}{2}x_4x_5^2, -x_3 + x_4x_5, -x_4, -x_5\right)$$

For $y \in R$

$$\pi_{\xi_1,\xi_3,\xi_4}(f)\phi(y) = \int_{R^5} f(x_1,x_2,x_3,x_4,x_5)\ \pi_{\xi_1,\xi_3,\xi_4}\left(-x_1 + x_2x_5 - \frac{1}{2}x_3x_5^2 + \frac{1}{6}x_4x_5^3, -x_2 + x_3x_5 - \frac{1}{2}x_4x_5^2,\right.$$

$$\left. -x_3 + x_4x_5, -x_4, -x_5\right)\phi(y)dx$$

$$= \int_{R^5} f(x)\exp\left[2\pi i\left\{-x_1 + x_2x_5 - \frac{1}{2}x_3x_5^2 + \frac{1}{6}x_4x_5^3 - \left(-x_2 + x_3x_5 - \frac{1}{2}x_4x_5^2\right)\right.\right.$$

$$\left.\left. y + \frac{1}{2}(-x_3 + x_4x_5)y^2 + \frac{1}{6}x_4y^3\right)\xi_1 + (-x_3 + x_4x_5 + x_4y)\xi_3 - x_4\xi_4\right\}\Big]\phi(y+x_5)dx$$

$$x_5 \to x_5 - y$$

$$= \int_{R^5} f(x_1,x_2,x_3,x_4,x_5 - y)\exp\left[2\pi i\left(-x_1 + x_2x_5 - x_2y - \frac{1}{2}x_3\left(x_5^2 + y^2 - 2x_5y\right)\right.\right.$$

$$+ \frac{1}{6}x_4\left\{(x_5 - y)^3 - \left(-x_2 + x_3x_5 - x_3y - \frac{1}{2}x_4(x_5 - y)^2\right)h + \frac{1}{2}(-x_3 + x_4x_5 - x_4y)\right.$$

$$\left.\left.\left. y^2 + \frac{1}{6}x_4y^3\right\}\xi_1 + (-x_3 + x_4x_5 - x_4y + x_4y)\xi_3 - x_4\xi_4)\right)\right]\phi(x_5)dx$$

$$= \int_{R^5} f(x_1,x_2,x_3,x_4,x_5 - y)\exp\left[2\pi i\left(-x_1 + x_2x_5 - \frac{1}{2}x_3(x_5 - y)^2 + 2(x_5 - y) + y^2\right)\right.$$

$$+\frac{1}{6}x_4\left\{(x_5-y)^3+3(x_5-y)^2y+3(x_5-y)y^2+y^3\right\}\xi_1-(x_3-x_4x_5)\xi_3-x_4\xi_4\Big)\Big]\,\phi\,(x_5)dx$$

$$=\int_{R^5} f(x_1,x_2,x_3,x_4,x_5-y)\exp\Big[-2\pi i\Big(x_1-x_2x_5+\frac{1}{2}x_3(x_5-y+y)^2$$

$$-\frac{1}{6}x_4(x_5-y+y)^3\Big)\xi_1+(x_3-x_4x_5)\xi_3+x_4\xi_4\Big]\phi(x_5)dx$$

$$=\int_{R^5} f(x_1,x_2,x_3,x_4,x_5-y)\exp$$

$$\left[-2\pi i\left((x_1-x_2x_5+\frac{1}{2}x_3x_5^2-\frac{1}{6}x_4x_5^3)\xi_1+(x_3-x_4x_5)\xi_3+x_4\xi_4\right)\right]\phi(x_5)dx$$

$$K^f_{\xi_1,\xi_3,\xi_4}(y,x_5)=\int_{R^5} f(x_1,x_2,x_3,x_4,x_5-y)\exp\left\{-2\pi i\left[\left(x_1-x_2x_5+\frac{1}{2}x_3x_5^2-\frac{1}{6}x_4x_5^3\right)\xi_1\right.\right.$$

$$\left.\left.+(x_3-x_4x_5)\xi_3+x_4\xi_4\right]\right\}dx_1dx_2dx_3dx_4$$

$$=F_{1234}f\left(\xi_1,-x_5\xi_1,\frac{1}{2}x_5^2\xi_1+\xi_3,-\frac{1}{6}x_5^3\xi_1,-x_5\xi_3+\xi_4,x_5-y\right)$$

$$\left\|\pi_{\xi_1,\xi_3,\xi_4}(f)\right\|^2_{HS}=\int_{R^2}\left|K^f_{\xi_1,\xi_2,\xi_4}(y,x_5)\right|^2dydx_5$$

$$=\int_{R^2}\left|F_{1234}f\left(\xi_1,-x_5\xi_1,\frac{1}{2}x_5^2\xi_1+\xi_3,-\frac{1}{6}x_5^3\xi_1-x_5\xi_3+\xi_4,x_5-y\right)\right|^2dydx_5$$

$$x_5\to\frac{-1}{\xi_1}x_5$$

$$=\frac{1}{|\xi_1|}\int_{R^2}\left|F_{1234}f\left(\xi_1,x_5,\frac{1}{2}\frac{x_5^2}{\xi_1}+\xi_3,\frac{1}{6}\frac{x_5^3}{\xi_1^2}+\frac{x_5x_3}{\xi_1}+\xi_4,\frac{-1}{\xi_1}x_5-y\right)\right|^2dydx_5$$

$$y\to-y-\frac{1}{\xi_1}x_5$$

$$=\frac{1}{|\xi_1|}\int_{R^2}\left|F_{1234}f\left(\xi_1,x_5,\frac{1}{2}\frac{x_5^2}{\xi_1}+\xi_3+\frac{1}{6}\frac{x_5^3}{\xi_1^2}+\frac{x_5x_3}{\xi_1}+\xi_4,y\right)\right|^2dydx_5$$

$$=\frac{1}{|\xi_1|}\int_{R^2}\left|F_{134}f\left(\xi_1,w\frac{1}{2}\frac{w^2}{\xi_1}+\xi_3,\frac{1}{6}\frac{w^3}{\xi_1^2}+\frac{w\xi_3}{\xi_1}+\xi_4,y\right)\right|^2dydw$$

REFERENCES

1. Folland, G., B., Sitaram, A. 1997. The uncertainity principle: *A mathematical survey*, J. Fourier analysis and application, **3**, 207-238.
2. Folland, G.B., 1995. *A course in Harmonic analysis,* CRC Press.
3. H. Rieter & J.D. Stegeman, 2000. *Classical Harmonic analysis and locally compact groups,* Oxford university press.
4. Hewitt, E., and Ross, K.A., 1963 & 1970. *Abstract Harmonic analysis* I & II, Springer Verlag.
5. Kaniuth, E.A., 2000. *Hardy theorem for simply connected nilpotent Lie groups*, Preprint.
6. Kumar, A., and Bhatta, C.R., 2004. *An uncertainty principle like Hardy's theorem for Nilpotent Lie groups*, J. Aust. Math. Soc, **76**, 1-7.
7. L. Corwin, L., and Greenleaf, F.P., 1990. *Representations of Nilpotent Liegroups and their applications*, Part I, *Basic theory and examples* (Cambridge University Press).
8. Ole, A. Nelson, 1983. *Uncertainty representations and coadjoint orbits of low-dimensional nilpotent Lie groups*, Queens Papers in pure and applied mathematics, No. 63.