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AN INTEGRAL TYPE COMMON FIXED POINT THEOREM IN DISLOCATED METRIC SPACE

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ABSTRACT

In this article, we establish a common fixed point theorem for two pairs of weakly compatible mappings with common limit range property in dislocated metric space.

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Keywords: Dislocated metric space, Common fixed point, Weakly compatible maps.

INTRODUCTION

In 1986, S. G. Matthews introduced the concept of dislocated metric space under metric domains in the context of domain theory. In 2000, P. Hitzler and A.K. Seda introduced the concept of dislocated topology where the initiation of dislocated metric space is appeared. Since then, many authors have established fixed point theorems in dislocated metric space. In the literature one can find many interesting articles in the field of dislocated metric space(See for examples [6, 7, 8, 9, 11, 12]). The study of fixed points of mappings satisfying a general contractive condition of integral type have been a very interesting and active field of research activity after the establishment of a theorem by A. Branciari [2]. The purpose of this article to establish a common fixed point theorem for weakly compatible mappings with common limit range property in dislocated metric space.

PRELIMINARIES

We start with the following definitions, lemmas and theorems.

Definition 1. [4]: *Let X be a non empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:*

1. $d(x, y) = d(y, x)$
2. $d(x, y) = d(y, x) = 0$ implies $x = y$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z $\in$ X.

Then d is called dislocated metric (or d-metric) on X and the pair (X, d) is called the dislocated metric space (or d-metric space).

Definition 2. [4]: A sequence $\{x_n\}$ in a d-metric space (X,d) is called a Cauchy sequence if for given $\varepsilon > 0$, there corresponds $n_0 \in \mathrm{N}$ such that for all $m,n \geq n_0$, we have $d(x_m, x_n) < \varepsilon$.

Definition 3. [4]: A sequence in d-metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

Definition 4. [4]: A d-metric space (X,d) is called complete if every Cauchy sequence in it is convergent with respect to d.

Lemma 1. [4]: Limits in a d-metric space are unique.

Definition 5: Let A and S be two self mappings on a set X. If $Ax = Sx$ for some $x \in X$, then x is called coincidence point of A and S.

Definition 6. [5]: Let A and S be mappings from a metric space (X,d) into itself. Then, A and S are said to be weakly compatible if they commute at their coincident point; that is, $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.

Definition 7. [13]: Let A and S be two self mappings defined on a metric space (X,d). We say that the mappings A and S satisfy (CLR_A) property if there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = Ax$$

MAIN RESULTS
Now we establish a common fixed point theorem for weakly compatible mappings using (CLR)-property.

Theorem 1: Let (X,d) be a dislocated metric space. Let $A, B, S, T : X \to X$ satisfying the following conditions

$$A(X) \subseteq S(X) \quad and \quad B(X) \subseteq T(X) \tag{1}$$

$$\int_0^{d(Ax,By)} \phi(t)dt \leq k \int_0^{M(x,y)} \phi(t)dt, \quad k \in [0, \frac{1}{2}) \tag{2}$$

where, $\qquad \phi : \mathrm{R}^+ \to \mathrm{R}^+$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \phi(t)dt > 0 \quad foreach \quad \varepsilon > 0 \tag{3}$$

$$M(x,y) = max\{d(Tx,Sy), d(Tx,Ax), d(By,Sy), \frac{1}{2}d(Tx,By), \frac{1}{2}d(Sy,Ax)\} \qquad (4)$$

1. The pairs (A,T) or (B,S) satisfy CLR-property

2. The pairs (A,T) and (B,S) are weakly compatible

Then,

the maps A and T have a coincidence point

the maps B and S have a coincidence point

the maps A, B, S and T have an unique common fixed point.

Proof: Assume that the pair (A,T) satisfy (CLR_A) property, so there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Tx_n = Ax \qquad (5)$$

for some $x \in X$. Since $A(X) \subseteq S(X)$, so there exists a sequence $\{y_n\} \in X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sy_n = Ax$. We show that

$$\lim_{n \to \infty} Bx_n = Ax \qquad (6)$$

From condition (2) we have,

$$\int_0^{d(Ax_n, By_n)} \phi(t)dt \le k \int_0^{M(x_n, y_n)} \phi(t)dt, \qquad (7)$$

where,

$$M(x_n, y_n) = max\{d(Tx_n, Sy_n), d(Tx_n, Ax_n), d(By_n, Sy_n),$$

$$\frac{1}{2}d(Tx_n, By_n), \frac{1}{2}d(Sy_n, Ax_n)\}$$

Taking limit as $n \to \infty$ we get,

$$\lim_{n \to \infty} \int_0^{d(Ax_n, By_n)} \phi(t)dt \le k \lim_{n \to \infty} \int_0^{M(x_n, y_n)} \phi(t)dt, \qquad (8)$$

Since,

$$\lim_{n\to\infty} d(Tx_n, Sy_n) = \lim_{n\to\infty} d(Tx_n, Ax_n) = \lim_{n\to\infty} d(Sy_n, Ax_n) = 0$$

$$\lim_{n\to\infty} d(Ax_n, By_n) = d(Ax, By_n) = \lim_{n\to\infty} d(Tx_n, By_n) = \lim_{n\to\infty} d(By_n, Sy_n)$$

Hence we have,

$$\int_0^{d(Ax, By_n)} \phi(t)dt \leq k \int_0^{d(Ax, By_n)} \phi(t)dt,$$

which is a contradiction, since $k \in [0, \frac{1}{2})$

Hence,

$$\lim_{n\to\infty} d(Ax, By_n) = 0, \text{ implies that } \lim_{n\to\infty} By_n = Ax.$$

Now we have,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sy_n = Ax$$

Assume $A(X) \subseteq S(X)$, then there exits $v \in X$ such that $Ax = Sv$.

We claim that $Bv = Sv$.

Now from condition (2)

$$\int_0^{d(Ax_n, Bv)} \phi(t)dt \leq k \int_0^{M(x_n, v)} \phi(t)dt, \tag{9}$$

where,

$$M(x_n, v) = max\{d(Tx_n, Sv), d(Tx_n, Ax_n), d(Bv, Sv),$$

$$\frac{1}{2}d(Tx_n, Bv), \frac{1}{2}d(Sv, Ax_n)\}$$

Since,

$$\lim_{n\to\infty} d(Tx_n, Bv) = d(Ax, Bv) = d(Sv, Bv)$$

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$$\lim_{n\to\infty} d(Tx_n, Sv) = \lim_{n\to\infty} d(Tx_n, Ax_n) = \lim_{n\to\infty} d(Sv, Ax_n) = 0$$

So, taking limit as $n \to \infty$ in (9) We conclude that

$$\int_0^{d(Sv,Bv)} \phi(t)dt \le k \int_0^{d(Sv,Bv)} \phi(t)dt, \qquad (10)$$

which is a contradiction. Hence $d(Sv, Bv) = 0$ *implies that* $Sv = Bv$. This proves that v is the coincidence point of of the maps B and S .

Hence, $Sv = Bv = Ax = w(Say)$

Since the pair (B, S) is weakly compatible, so

$BSv = SBv$ *implies that* $Bw = Sw$

Since $B(X) \subseteq T(X)$ there exists a point $u \in X$ such that $Bv = Tu$. We show that

$Tu = Au = w$

From condition (2),

$$\int_0^{d(Au,Bv)} \phi(t)dt \le k \int_0^{M(u,v)} \phi(t)dt,$$

where,

$$M(u,v) = max\{d(Tu, Sv), d(Tu, Au), d(Bv, Sv),$$

$$\frac{1}{2}d(Tu, Bv), \frac{1}{2}d(Sv, Au)\}$$

$$= max\{d(Bv, Bv), d(Bv, Au), d(Bv, Bv),$$

$$\frac{1}{2}d(Bv, Bv), \frac{1}{2}d(Bv, Au)\}$$

$$= max\{d(Bv, Bv), d(Bv, Au)\}$$

Hence,

$$\int_0^{d(Au,Bv)} \phi(t)dt \le k \int_0^{max\{d(Bv,Bv),d(Bv,Au)\}} \phi(t)dt,$$

Since,

$$d(Bv, Bv) \leq 2d(Au, Bv)$$

So if $\max\{d(Bv, Bv), d(Bv, Au)\} = d(Bv, Bv)$ or $d(Bv, Au)$ we get the contradiction for both cases

Therefore $d(Au, Bv) = 0$ *implies that* $Au = Bv$.

$$\therefore Au = Bv = Tu = w$$

This proves that u is the coincidence point of the maps A and T.

Since the pair (A, T) is weakly compatible so,

$$ATu = TAu \; implies \; that \; Aw = Tw$$

We show that $Aw = w$.

From condition (2),

$$\int_0^{d(Aw, w)} \phi(t)dt = \int_0^{d(Aw, Bv)} \phi(t)dt \leq k \int_0^{M(w,v)} \phi(t)dt,$$

where,

$$M(w, v) = max\{d(Tw, Sv), d(Tw, Aw), d(Bv, Sv),$$

$$\frac{1}{2}d(Tw, Bv), \frac{1}{2}d(Sv, Aw)\}$$

$$= max\{d(Aw, w), d(Aw, Aw), d(w, w),$$

$$\frac{1}{2}d(Aw, w), \frac{1}{2}d(w, Aw)\}$$

$$= max\{d(Aw, w), d(Aw, Aw)d(w, w)\}$$

Since,

$$d(Aw, Aw) \leq 2d(Aw, w) \quad and \quad d(w, w) \leq 2d(Aw, w)$$

So if $\max\{d(Aw, w), d(Aw, Aw), d(w, w)\} = d(Aw, w)$ or $d(Aw, w)$ or $d(w, w)$ we have,

$$\int_0^{d(Aw,w)}\phi(t)dt \le \int_0^{d(Aw,w)}\phi(t)dt \quad or \quad \le 2k\int_0^{d(Aw,w)}\phi(t)dt$$

which give contradictions for all three cases.

Hence $d(Aw,w)=0$ *implies that* $Aw=w$. Similarly we obtain $Bw=w$.

$\therefore Aw = Bw = Sw = Tw = w$. Hence w is the common fixed point of four mappings A,B,S and T.

Uniqueness:

Let $z(\ne w)$ be other common fixed point of the mappings A,B,S and T, then by the condition (2)

$$\int_0^{d(w,z)}\phi(t)dt = \int_0^{d(Aw,Bz)}\phi(t)dt \le k\int_0^{M(w,z)}\phi(t)dt \tag{11}$$

where,

$$M(w,z) = max\{d(Tw,Sz),d(Tw,Aw),d(Bz,Sz),$$

$$\frac{1}{2}d(Tw,Bz),\frac{1}{2}d(Sz,Aw)\}$$

$$= max\{d(w,z),d(w,w),d(z,z),$$

$$\frac{1}{2}d(w,z),\frac{1}{2}d(z,w)\}$$

$$= max\{d(w,z),d(w,w),d(z,z)\}$$

Since,

$$d(w,w) \le 2d(w,z) \quad and \quad d(z,z) \le 2d(z,w)$$

So if $max\{d(w,z),d(w,w),d(z,z)\} = d(w,z)$ or $d(w,w)$ or $d(z,z)$ we have

$$\int_0^{d(w,z)}\phi(t)dt = \int_0^{d(Aw,Bz)}\phi(t)dt \le k\int_0^{M(w,z)}\phi(t)dt \le k\int_0^{d(w,z)}\phi(t)dt$$

or,

$$\int_0^{d(w,z)}\phi(t)dt \le 2k\int_0^{d(w,z)}\phi(t)dt$$

which give contradictions for all three cases.

Hence, $d(w, z) = 0$ *implies that* $w = z$. This establishes the uniqueness of the common fixed point.

Now we have the following corollaries:

If we put $T = S$ in theorem (1) we obtain the following corollary.

Corrollary 1: *Let (X,d) be a dislocated metric space. Let* $A, B, S : X \rightarrow X$ *satisfying the following conditions*

$$A(X), B(X) \subseteq S(X) \tag{12}$$

$$\int_0^{d(Ax,By)} \phi(t)dt \le k \int_0^{M(x,y)} \phi(t)dt, \quad k \in [0, \frac{1}{2}) \tag{13}$$

where, $\phi : \mathsf{R}^+ \rightarrow \mathsf{R}^+$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \phi(t)dt > 0 \quad foreach \quad \varepsilon > 0 \tag{14}$$

$$M(x, y) = max\{d(Sx, Sy), d(Sx, Ax), d(By, Sy), \frac{1}{2}d(Sx, By), \frac{1}{2}d(Sy, Ax)\} \tag{15}$$

1. The pairs (A, S) or (B, S) satisfy CLR-property

2. The pairs (A, S) and (B, S) are weakly compatible

Then,

the maps A and S have a coincidence point

the maps B and S have a coincidence point

the maps A, B and S have an unique common fixed point.

If we put $B = A$ in theorem (1) we obtain the following corollary.

Corrollary 2: Let (X,d) be a dislocated metric space. Let $A, S, T : X \rightarrow X$ satisfying the following conditions,

$$A(X) \subseteq S(X) \quad and \quad A(X) \subseteq T(X) \tag{16}$$

$$\int_0^{d(Ax,Ay)} \phi(t)dt \le k \int_0^{M(x,y)} \phi(t)dt, \quad k \in [0, \frac{1}{2}) \tag{17}$$

where, $\quad \phi: R^+ \to R^+$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \phi(t)dt > 0 \quad foreach \quad \varepsilon > 0 \tag{18}$$

$$M(x,y) = max\{d(Tx,Sy), d(Tx,Ax), d(Ay,Sy), \frac{1}{2}d(Tx,Ay), \frac{1}{2}d(Sy,Ax)\} \tag{19}$$

1. The pair (A,T) and (A, S) satisfy CLR-property

2. The pairs (A,T) and (A,S) are weakly compatible

Then,

the maps A and T have a coincidence point

the maps A and S have a coincidence point

the maps A, S and T have an unique common fixed point.

If we put $T = S$ and $B = A$ in theorem (1) we obtain the following corollary.

Corrollary 3: Let (X,d) be a dislocated metric space. Let $A, S : X \to X$ satisfying the following conditions

$$A(X) \subseteq S(X) \tag{20}$$

$$\int_0^{d(Ax,Ay)} \phi(t)dt \le k \int_0^{M(x,y)} \phi(t)dt, \quad k \in [0, \frac{1}{2}) \tag{21}$$

where, $\quad \phi: R^+ \to R^+$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \phi(t)dt > 0 \quad foreach \quad \varepsilon > 0 \tag{22}$$

$$M(x, y) = max\{d(Sx, Sy), d(Sx, Ax), d(Ay, Sy), \frac{1}{2}d(Sx, Ay), \frac{1}{2}d(Sy, Ax)\} \tag{23}$$

1. The pair (A, S) satisfy CLR-property

2. The pair (A, S) is weakly compatible

Then,

the maps A and S have a coincidence point

the maps A and S have an unique common fixed point.

If we put $T = S = I(Identity\ map)$ the we get the following corollary.

Corrollary 4: Let (X,d) be a dislocated metric space. Let $A, B, I : X \rightarrow X$ satisfying the following conditions

$$A(X), B(X) \subseteq I(X) \tag{24}$$

$$\int_0^{d(Ax, By)} \phi(t)dt \leq k \int_0^{M(x,y)} \phi(t)dt, \quad k \in [0, \frac{1}{2}) \tag{25}$$

where, $\phi : \mathsf{R}^+ \rightarrow \mathsf{R}^+$

is a Lebesgue integrable mapping which is summable, non-negative and such that

$$\int_0^\varepsilon \phi(t)dt > 0 \quad foreach \quad \varepsilon > 0 \tag{26}$$

$$M(x, y) = max\{d(x, y), d(x, Ax), d(By, y), \frac{1}{2}d(x, By), \frac{1}{2}d(y, Ax)\} \tag{27}$$

1. The pairs (A, I) or (B, I) satisfy CLR-property

2. The pairs (A, I) and (B, I) are weakly compatible

Then,

the maps A, B and I have an unique common fixed point.

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